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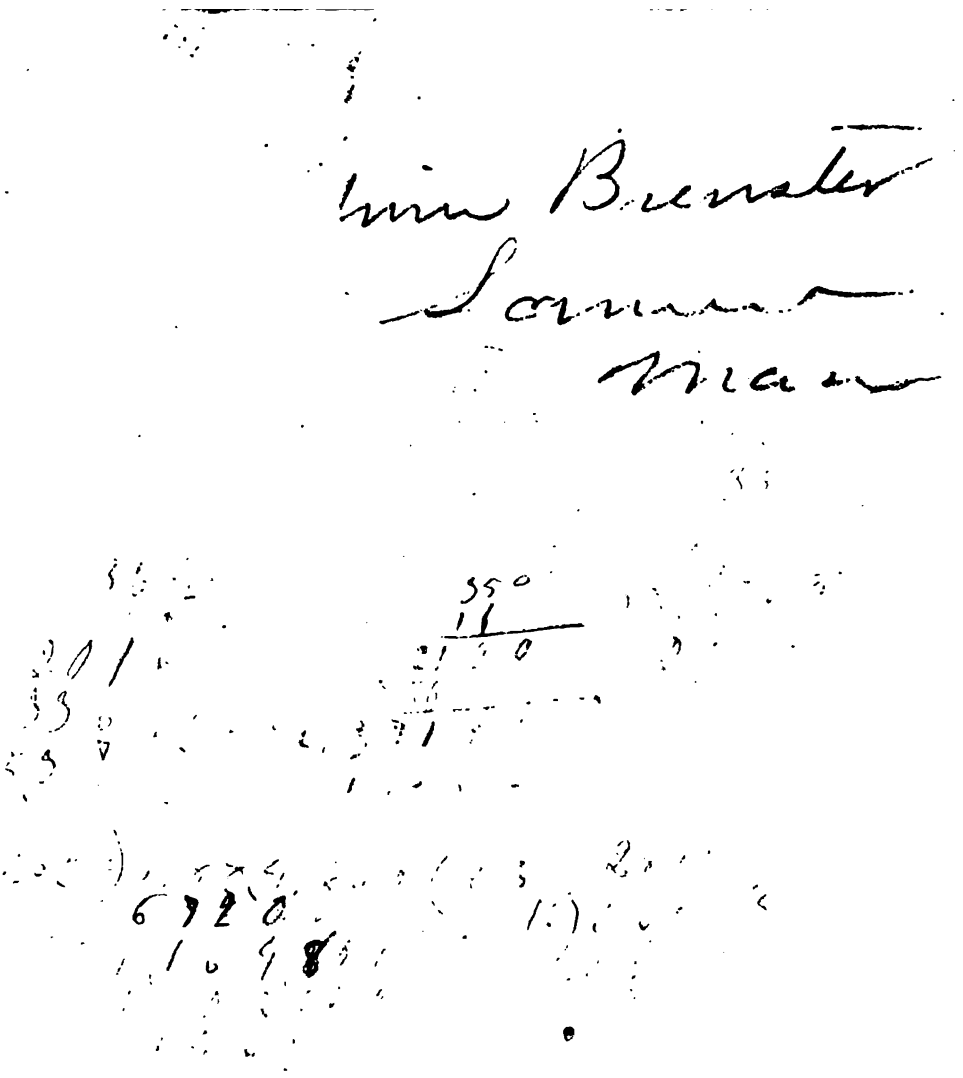
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AN

INTRODUCTION

TO

NATURAL PHILOSOPHY:

DESIGNED AS A

TEXT BOOK,

FOR

THE USE OF THE STUDENTS IN

YALE COLLEGE.

IN TWO VOLUMES.

VOL. I.—MECHANICS AND HYDROSTATICS.

COMPILED FROM VARIOUS AUTHORITIES.

BY DENISON OLMSTED, A. M.
PROFESSOR OF NATURAL PHILOSOPHY AND ASTRONOMY.

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PREFACE.

THE compiler of this work has had two objects constantly in view; first, to make the student thoroughly and familiarly acquainted with the *leading principles* of Natural Philosophy; and secondly, to furnish him with as much *useful information* as possible, within so limited a compass. The foundation of all accurate attainments in Natural Philosophy and Astronomy being laid in the science of Mechanics, a large proportion of the work is devoted to this subject. The Mathematical Elements of Mechanics are moreover first considered, separately, that the student may have nothing to divert his mind from the contemplation of these universal and fundamental truths. Still farther to render the knowledge of these truths familiar, as well as to supply a most useful intellectual exercise to the student, a great variety of Problems are annexed to each chapter, the utility of which must be obvious to every experienced instructor. Indeed, Problems hold so important a place in the estimation of the writer, that he has introduced them into various parts of the work, wherever the subject appeared to be susceptible of deriving aid from them. Problems put the student upon his own resources; they compel him to think for himself; they lead him to a just understanding of the principles demonstrated; and they teach him how to reduce his knowledge to practice. These truths are so obvious, that it is difficult to account for the singular fact, that treatises on Natural Philosophy have, in general, contained few or no problems, although they occupy so large a space in most of the branches of the pure mathematics.

The first part of the following treatise on Mechanics, comprising the "Mathematical Elements," is taken chiefly from *Bridge's Mechanics*. This work was peculiarly adapted to our purpose, partly because it is written in a style well suited to the average capacities and attainments of college classes, and partly because it is enriched with a finer collection of problems than any similar work with which we are acquainted. We have aimed to select such parts as promised the most practical utility; and in order to adapt the treatise to the purposes of recitation, the propositions are more distinctly enunciated

than in the original work, and various alterations, and occasional additions, are introduced, and explanations added by note or otherwise, with the view of suiting it better to the course of instruction adopted in Yale College.

In Part II, the *Practical Applications of the Principles of Mechanics to the Arts and to the Phenomena of Nature*, are pursued as far as our limits would permit, and farther perhaps than some instructors will deem necessary; for we are aware that some maintain the expediency of occupying the attention of the student almost exclusively with *general principles*, and leaving him to make the application for himself. According to our experience, however, the student who is furnished with the knowledge of abstract principles merely, seldom acquires the necessary readiness in reducing them to practice. It has appeared to us no less necessary to initiate the learner in the habit of philosophizing, than in the doctrines of philosophy. In this manner, he will, indeed, acquire the knowledge of fewer principles; but he will know much better how to use his acquisitions.

We cannot, however, agree with those instructors who have yielded to the spirit of the age, (which is still hunting after a "royal road" to knowledge,) so far as to forsake demonstration altogether, and substitute for the mathematical elements of Natural Philosophy, text books in which the principles of the science rest on no better basis than mere popular illustration. Works of this kind may, indeed, furnish us with very useful materials for exhibiting the *applications* of these principles. The most successful writer of this class is *Dr. Lardner*, whose treatise on Mechanics, in particular, is an elegant specimen of a popular work on science. We have accordingly made frequent use of that work in the practical part of Mechanics. In the chapter on Machinery, we have also several times had recourse to the excellent popular treatise of *Dr. Bigelow*, on the "Elements of Technology." *Dr. Arnott*, in his "Elements of Physics," has afforded many felicitous illustrations of philosophical truths. These we have occasionally adopted; but the entire work is too diffuse, and too barren of principles, to be recommended to the student of philosophy. The philosophical treatises in the Library of Useful Knowledge, have a higher degree of merit; although some of them are too superficial for the scholar, and others too recondite for the general reader.

The subsequent parts of the work are compiled from various authors. In the belief that the truths most important to be inserted in a text book on Natural Philosophy, are, in general, such as have long been known, no effort has been made to conform the style, either of the propositions or the demonstrations, to a modern dress; but authors, old and new, English and French, have been consulted and used indiscriminately. Aiming solely at preparing such an elementary work, as would be most useful to the academic student, we have not aspired to the praise of originality, nor felt at liberty to consult for the pride of authorship.

Since the publication of the first edition of this work, the compiler has been favored with the opinions of a number of his brethren of sister Colleges, who have used it as a Text Book. A few would prefer to have a more strictly mathematical complexion preserved throughout, in the place of those parts which are written in a more popular style, since the method of expressing philosophical truths by mathematical formulæ, is more concise and comprehensive, and better suited to mental discipline, and the cultivation of a mathematical taste. A greater number, however, of those who have been so good as to communicate their views to the writer, have approved of the method adopted,—namely, to establish the *general principles* of the science by rigorous mathematical demonstration, or by precise experiments, but to rehearse the *applications* of those principles to the arts, and to the phenomena of nature, in a style divested, as far as possible, of technical phraseology. They believe that such a method renders the study of Natural Philosophy peculiarly attractive to the young learner, and conducive to the formation of habits of philosophical observation, while they rely more on other parts of the scientific course, particularly on the pure mathematics, to fulfil the purposes of mental discipline, and to inspire a mathematical taste. We have desired to accomplish as far as possible, these several objects of a philosophical education,—to improve the faculties of the mind itself, to imbue it with a love of rigorous demonstration, and to commence the formation of habits of philosophical observation, which shall be carried forward, beyond the pale of academic study, to be confirmed and strengthened throughout the period of after life. The variety of subjects comprehended under Natural Philosophy, some admitting of strict geometrical and analytical reasoning, and others con-

ducted wholly by experimental research, is well adapted to the attainment of these important objects; and it is the prerogative of this science, at once to enlarge the mind by the most profound inquiries, and to conduct it through the most delightful and varied fields of experiment and observation.

In a work necessarily so limited as this, (when compared with the vast extent of the subject,) many topics must be treated with extreme conciseness, and many others, essential to a complete philosophical education, must be omitted altogether. While we aim to furnish the student with a knowledge of the *great laws of nature*, and to exemplify and illustrate them by numerous applications, we can claim nothing more than an "Introduction to Natural Philosophy," suited to beginners; while we take the liberty of recommending to the more advanced student, the Treatise of Professor Renwick on Mechanics,—a work containing a full exhibition of the principles of the science, and enriched with a great amount of practical information.

It is recommended to the student to make a free use of the *Analysis* prefixed to each volume, especially in reviewing. Let him submit each of the particulars indicated in this outline, to deliberate and repeated reflection, and he will not only fully possess himself of the contents of the work, but will lay up in the mind a system of heads, or "common places," under which he can conveniently and usefully arrange all his future acquisitions on similar subjects.

The course of instruction in Natural Philosophy pursued in Yale College, to which this treatise is adapted, proceeds as follows.

The *mathematical* part of mechanics is first recited, in the same manner as a branch of the pure mathematics. With the *practical* part commences a series of familiar Lectures,* designed to amplify the text, and to illustrate it by numerous experiments. These are continued during the perusal of the remainder of the work.

To the same class is afterwards delivered a course of *written* Lectures, which are chiefly devoted to the discussion of the great principles of Philosophy and Astronomy, and especially to such

* Should any teacher who may use this work, think it better to connect the Lectures and Practical Applications immediately with the theoretical part, it will be easy to do so, by giving occasional lessons in Part II, while the student is reading Part I.

subjects as require a fuller attention than they can receive in the elementary course.

In the present edition, great pains has been taken to insure accuracy. The whole work has been thoroughly revised, and some parts re-written. To many of the problems, new conditions are annexed. This became necessary, by the publication of "solutions" of the problems contained in the previous editions. It is also desirable, in a public Institution, that each class, in solving problems, should be entirely independent of the labors of preceding classes. For the chapter on Projectiles, in the previous editions, we have substituted a new article, in which the *analytical method* is exemplified. To such a subject this method is peculiarly adapted; although we do not think it so well suited to promote the great objects of a liberal education, as that adopted in the treatise which forms the basis of the first part of this work.

Yale College, March, 1838.

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ERRATA FOR VOL. I.

Page 33, Ans. to Quest. 13, for 96.96, *read* 30.476.
" 86, Art. 92, for $(1 \times r)$ *read* $(1 + r)$.
" 121, for P, *read* p.
" 178, for $(4 + 6400)$, *read* $(4L + 6400)$.
" 231, line 4, *dele* to.
" 234, Art. 287, for 92 and 94, *read* 82 and 84.
" 340, for Art. 357, *read* Art. 354.

☞ As many of the figures are too difficult of construction, to admit of being drawn during the recitations, the philosophical department of Yale College, is furnished with a set of *Cards*, on which the diagrams are drawn, on a scale sufficiently large, to be distinctly seen in all parts of the lecture-room. The cards are furnished by the bookbinder, and consist of a sheet of pasteboard about eighteen inches square, on both sides of which is pasted white paper suitable for drawing, and a finish and stability are given to the card by binding the edges with colored paper. A string of tape is attached to the top, and the card is hung up and used like the black board. The *lettering*, (which would be tedious and laborious with a pen,) is best done by pasting on large printed letters.

PART I.—MATHEMATICAL ELEMENTS OF MECHANICS.

[ON THE BASIS OF BRIDGE'S MECHANICS.]

PRELIMINARY DEFINITIONS.

ART. 1. *Natural Philosophy* is the science which treats of the Laws of the Material world.*

The term *law*, as here used, signifies *the mode in which the powers of nature act.* Laws are *general truths*, comprehending a great number of subordinate truths.

Natural or Mechanical Philosophy is divided into Mechanics, Hydrostatics, Pneumatics, Electricity, Magnetism, and Optics.

MECHANICS *is that branch of Natural Philosophy, which treats of the Equilibrium and Motion of bodies.* As the changes which occur between masses of matter, involve the idea of motion, hence, the causes which produce motion, or which prevent it, and the manner in which it takes place, (its *laws*,) constitute the great object of inquiry in mechanical philosophy.

Body is any collection of matter existing in a separate form.

Force is any cause which moves or tends to move a body, or which changes or tends to change its motion. Every force produces *actual* motion if it is not counteracted by contrary forces; but if it remains counteracted, the motion which it *tends* to produce is called *Virtual.*

That part of Mechanics which relates to the action of forces producing *equilibrium* or *rest*, in bodies, is called *Statics;* that which relates to the action of forces producing *motion*, is called *Dynamics.*†

* This is the primitive signification of the term Natural Philosophy; but the vast extension given to inquiries into the laws of nature, rendered a division of them necessary. Hence, those laws of nature which relate to *masses* of matter were retained by Natural Philosophy, (which has been farther divided into Mechanical Philosophy and Astronomy,) while those which relate to *particles* of matter, and to the changes of constitution produced by their action on each other, were assigned to Chemistry.

† In the following treatise, it is found convenient to disregard this distinction.

The science of Mechanics comprehends those laws of equilibrium and motion only, which are common to all bodies in the universe, and to bodies in every form, whether solid, fluid or äeriform; but the laws of equilibrium and motion undergo certain additional modifications in consequence of the peculiar properties of *fluids*. Hence, that branch of Mechanics which treats of the peculiar mechanical properties of fluids in the form of water, is called *Hydrostatics*; and that which treats of the peculiar mechanical properties of fluids in the form of air, is called *Pneumatics*.

2. The two *essential* properties of matter, both of which are inseparable from it, are *extension* and *impenetrability*. Extension, in the three dimensions of length, breadth, and thickness, belongs to matter under all circumstances; and impenetrability, or *the property of excluding all other matter from the space which it occupies*, appertains alike to the largest body and to the smallest particle.

The word *particle* is much used in writings on physical subjects. In Natural Philosophy we mean by particles, the *smallest parts* into which a body may be supposed to be divided by mechanical means, without any reference to the different elements of which such particles may be composed. Inquiries of this nature belong to Chemistry.

The quantity of matter which a body contains, is called its *Mass*; the space it occupies, its *Volume*; its relative quantity of matter under a given volume, its *Density*. All bodies have empty spaces denominated *pores*. In solids, we may often see the pores with the naked eye, and almost always by the microscope; in fluids, their existence can be proved by experiment. The ratio of the space occupied by the pores of a body to that occupied by the solid matter, is not known; but there are reasons for believing that, even in the densest bodies, the amount of solid matter is small compared with the empty spaces.* Hence it is inferred that the particles of matter touch each other only in a few points.†

Although extension and impenetrability are said to be the *essential* properties of matter, because they are inseparable from its very existence, yet there are also several other properties which are known by experience to belong to all matter, as *gravity*, *inertia*,

* See Newton's Optics, Lib. II. iii. Pr. 8. † Playfair.

and *divisibility;* and others still which belong not to matter universally, but only to certain classes of bodies, as *elasticity*, or the power a body has of recovering itself when compressed; *malleability*, or the power of being extended into leaves or plates; and *ductility*, or the power of being extended in length, as when drawn into wire.

In *Geometry*, we conceive figures to possess extension only without solidity; or to occupy space without excluding other figures from it; but in *Mechanics*, we take objects such as they occur in nature, viz. not only extended, but *impenetrable.**

CHAPTER I.

OF MOTION AND THE LAWS OF MOTION.

3. ATTRACTION is the tendency which one portion of matter has towards another, and exists both between *particles* and between *masses* of matter. *Aggregation* is the union of particles of the *same kind* in one body; as of the particles of lead in a musket ball. *Affinity* is the union of particles of *different kinds* in one body; as of particles of copper and of zinc in brass. *Cohesion* is the union of *compound* particles in one body; thus, a particle of copper and a particle of zinc are united to form a particle of brass by affinity, and the particles of brass are united by cohesion. In mechanical philosophy, however, the term cohesion is usually employed to denote the union of particles of all sorts whether simple or compound, leaving to Chemistry all inquiries respecting the composition of bodies.

Gravity is that property by which all terrestrial bodies tend towards the center of the earth. It is in this sense that gravity is understood as a force in Mechanics. But in order to give the learner correct views of this important subject, we subjoin a few other particulars respecting it.

4. *Gravity is a property of matter, universally; and the force of gravity in any body is proportioned to its quantity of matter.*

* Whewell's Mechanics, p. 2.

Since every particle of matter is endowed with this property, it follows that the force of gravity is proportioned to the mass or quantity of matter.* We do not say what gravity *is*, but what it *does*,—namely, that it is something which gives to every particle of matter a tendency towards every other particle. This influence is conveyed from one body to another without any perceptible interval of time.†

Gravity extends to all known bodies in the universe, from the smallest to the greatest; but the consideration of the subject, in this extent, belongs to astronomy. We at present contemplate gravity only as it affects *terrestrial* bodies. By it all bodies are drawn towards the center of the earth, not because there is any peculiar property or power in the center, but because, the earth being a sphere, the *aggregate* effect of the attractions exerted by all its parts upon any body exterior to it, is such as to direct the body towards the centre; as will be more fully explained hereafter.

5. This property discovers itself, not only in the motion of falling bodies, but in the *pressure* exerted by one portion of matter upon another which sustains it; and bodies descending freely under its influence, whatever be their figure, dimensions, or texture, are all *equally accelerated* in right lines perpendicular to the plane of the horizon. The apparent *inequality* of the action of gravity upon different species of matter near the surface of the earth, arises entirely from the resistance which they meet with in their passage through the air. When this resistance is removed, (as in the exhausted receiver of an air-pump,) no such inequality is perceived; bodies of all kinds there descend with equal velocities; and a guinea, a feather, and the smallest particle of matter, if let fall together, are observed to reach the bottom of the receiver exactly at the same instant.

6. The *weight* of a body is the force it exerts in consequence of its gravity, and is measured by its mechanical effects, such as bend-

* A decisive proof that the force of gravity is always proportioned to the quantity of matter, is furnished by the pendulum, its vibrations (which depend on gravity and measure its force, as will be shown hereafter) being always performed in the same time, of what material soever it is made.—*Francœur*, Mech. p. 63.

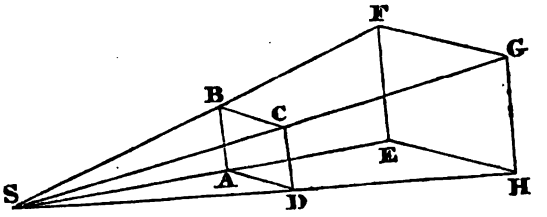
† If the action of gravitation is not instantaneous, it moves more than fifty millions of times faster than light.—*Fourier*, Eulogy on La Place.

ing a spring, or turning a balance. The force thus exerted by a given mass of matter, (as a cubic foot of water,) being taken as the standard, called 1000, and accurately counterpoised in a balance by some substance easily susceptible of division, (as a mass of lead for example,) multiples or aliquot parts of this standard weight, afford the means of estimating the weights of all other bodies. Hence, *weights* are nothing more than *measures of the force of gravity* in different bodies; but since the force of gravity is proportioned to the quantity of matter, (Art. 4,) weights are also measures of the comparative quantities of matter in different bodies.

7. *Gravity at different distances from the Earth, varies inversely as the square of the distance from its center.*

The total amount of attraction exerted by the earth upon bodies exterior to it, is the same as though that force were all concentrated in the center. (Art. 4.) But a force or influence which proceeds in right lines from a point in every direction, is diminished as the square of the distance is increased. For,

Fig. 1.



Let S be the center of the earth; and since the force of gravity acts in right lines directed towards that center, whatever be the nature of gravity, its influence at the distance SA, will be equally diffused over the surface ABCD; and, at the distance SE, it will be equally diffused over the surface EFGH. Therefore its intensity or force will be as much less at the point E than at A, as EFGH is greater than ABCD; that is, the force of gravity at E, is to the force of gravity at A, as EFGH to ABCD inversely.

But EFGH and ABCD, being similar figures,* are to one another as the squares of their homologous sides, that is, as EF^2 to AB^2. ∴ the forces of gravity at E and A, are inversely as $EF^2 : AB^2$. Again,

* Euc. 12. 3. Sup.

since SAB and SEF are similar triangles, $EF^2 : AB^2 :: SE^2 : SA^2$. ∴

The forces of gravity at E and A are inversely as $SE^2 : SA^2$.—Q. E. D.

8. The *weight* of a body, therefore, will vary at different heights above the earth's surface. Thus, at twice the distance from the center, or at the height of about 4000 miles above the earth, the force of gravity is only one fourth as great as at the surface, and a given body would weigh only one fourth as much as at the earth. The moon being 60 times as far from the earth's center, as the distance from that center to the surface, the attraction of the earth upon the moon is 3600 (60^2) times less than upon bodies near the earth. But the heights at which experiments are commonly made upon the weights of bodies, bear so small a ratio to the radius of the earth, that this variation is commonly imperceptible. At the height of *half a mile*, the diminution does not amount to more than about $\frac{1}{4000}$th part of the weight at the surface.

For, let r=the radius of the earth=4000 miles; and let x be the height of the body, W its weight at the earth's surface, and W′ its weight at the height x. Then

$$W : W' :: \overline{r+x}^2 : r^2 :: r^2+2rx+x^2 : r^2.$$

But when x is a small fraction of r,* x^2 may be neglected, and then

$$W : W' :: r^2+2rx : r^2 :: r+2x : r,$$

$$W : W-W' :: r+2x : 2x. \therefore$$

$W-W' = \frac{W \times 2x}{r+2x}$ = the difference between the weights at the surface of the earth and at the height denoted by x. Let x be *half a mile;* then this difference $= \frac{W \times 1}{4000+1} = \frac{1}{4001}$th part of the whole weight. But if the height were 100 miles above the earth, then $\frac{100}{4000} = \frac{1}{40}$; and the square of this $= \frac{1}{1600}$ of the radius of the earth, a quantity too large to be neglected, and the difference of weights at the surface and at the height of 100 miles, will be found by the formula $\frac{W \times 2rx+x^2}{r^2+2rx+x^2}$, derived from the first proportion.

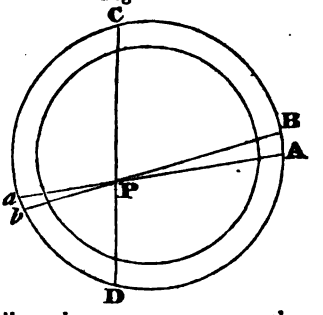
* If, for example, $x = \frac{1}{8000}$ of r, or $\frac{1}{2}$ a mile, then $x^2 = \left(\frac{1}{8000}\right)^2 = \frac{1}{64,000,000}$, a quantity so small that it may be neglected.

What loss of weight would a body sustain by being elevated 500 *miles above the earth?* Ans. $\frac{1}{17}$. Were x^2 neglected, then the loss would be $\frac{1}{18}$, which differs from $\frac{1}{17}$ by only $\frac{1}{306}$.

9. *A body situated within a hollow sphere, would remain at rest in any part of the void.**

The force of gravity is directly as the quantity of matter, and inversely as the square of the distance; that is, $G : g :: \frac{Q}{D^2} : \frac{q}{d^2}$; or $G \propto \frac{Q}{D^2}$.† (Art. 4 and 7.)

Fig. 2.



Let CAD represent the surface of a hollow sphere, and P any point in the void. Through P let the plane CD pass, dividing the sphere into any two segments CAD and C*a*D.

Let PBA and P*ba* represent two cones meeting in a very small angle at P, and having their bases in the surface of the sphere; which bases, being indefinitely small, may be considered as plane figures, and, being similar, they are to one another as the squares of their homologous sides; that is, as AB^2 to ab^2. But when BA and *ba* are indefinitely small, the lines PB and PA may be considered equal, as also P*a* and P*b*. Therefore, the two triangles PAB and P*ab* are similar, and $AP^2 : aP^2 :: AB^2 : ab^2$. But $AB^2 : ab^2 ::$ base of PAB : base of P*ab* $:: Q : q$‡. $\therefore AP^2 : aP^2 :: Q : q :: D^2 : d^2$.§ Hence $\frac{Q}{D^2} = \frac{q}{d^2}$. But, as before shown, $G : g :: \frac{Q}{D^2} : \frac{q}{d^2}$. $\therefore G = g$; that is, the point P (or a body at P) is equally attracted towards AB and *ab*, and the same will be the case with all the corresponding portions of the two opposite segments.

* The solid part of the sphere is supposed to be throughout of uniform density.

† See the doctrine of "Variation or General Proportion," as explained in Day's Algebra, Art. 403, &c. It is earnestly recommended to the learner to refresh his memory on this subject, great use being made of it in Mechanics.

‡ The surface of the sphere is here considered as a thin lamina of matter.

§ Since here D and d are the same as AP and aP.

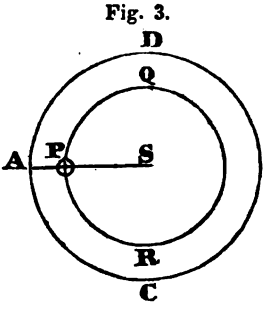
The same reasoning evidently applies to all the concentric surfaces or laminæ of which the shell of the sphere may be supposed to be made up; therefore *a body situated within a hollow sphere, &c.**

10. *The force of gravity below the earth's surface is, at different distances from the center, directly as those distances.*

Were a body placed at the center of the earth, being attracted equally in all directions, it would evidently remain at rest; and were it situated at any point between the center and the surface, the force of gravity towards the center would be diminished by the loss of the attraction of the matter exterior to it; for the matter exterior to it would, by Article 9, have no effect upon it. Thus, if a body were to fall through a hole bored from the earth's surface to its center, the gravity of it would constantly diminish, until, at the center, it would become nothing.

Let P be a body situated within the earth at any distance (D) from the center. Then it appears by the last article, that the gravity of P towards the center would not be affected at all by the shell exterior to PQR, and that P would gravitate only by the force exerted by the sphere PQR. But this force (G) is directly as the quantity of matter in PQR, and inversely as the square of the distance from the center. That is, $G \propto \frac{Q}{D^2}$. But $Q \propto D^3$†. $\therefore G \propto \frac{D^3}{D^2}$ $\propto D$. Therefore, the gravity of P varies as its distances from the center of the sphere.‡

Fig. 3.



11. The Inertia of matter is its *resistance to a change of state, whether of rest or motion.* The inertia of a body at rest is the resistance to be overcome to bring it to a given velocity; or, in common language, "to start it;" and the inertia of a body in motion, is the resistance it makes to being stopped, after the moving force is with-

* See Newton's Principia, Book I, Pr. 70.

† The quantity of matter in spheres being as the cubes of their radii.

‡ Principia, Book I, Pr. 73.

drawn. Thus the inertia of a steamboat, while getting under weigh, requires a great expenditure of force to bring the boat to its final velocity; but its inertia carries it still forward after the engine is stopped. Since every particle is endued with this property, *the inertia of a body is proportioned to its quantity of matter, and of course* (Art. 6,) *to its weight.* Inertia however is a more sure criterion of the quantity of matter in a body than weight is; for inertia is always, and under all circumstances, the same; whereas weight, being merely the measure of gravity, is diminished as gravity is diminished; so that it is less on the tops of high mountains, than at the general level of the earth.

12. In observing the phenomena* connected with the actual motion of a body, we consider the *space* over which it moves, the *time* of its motion, and the *velocity.*

A body is said to move with *uniform* velocity when *it describes equal spaces in equal times.* When the spaces described by it in equal portions of time continually increase, it is said to move with an *accelerated* velocity; and with a *retarded* velocity, when those spaces continually decrease. If its motion is so regulated, that it receives equal increments of velocity in equal times, then it is said to be *uniformly accelerated;* and *uniformly retarded,* if the body suffers equal decrements of velocity in equal times.

The space described by a body moving with uniform velocity, increases in the compound ratio of the time and velocity.

For, a body moving 10 seconds, at the rate of 40 feet per second, will move over 10×40, or 400 feet; and let x=the number of seconds for which a body moves uniformly, and let y=the number of feet described in each second; then it is evident that xy will denote the number of feet (i. e. the space) described by it in x seconds.

In general, if S=the space described by a body; T=the time of its motion, expressed in seconds; V= the uniform velocity with

* The word *phenomena* is much used in Natural Philosophy. It is thus defined: The phrase Natural Phenomena, in its widest acceptation, denotes any effects in the material part of the creation, addressed to one or more of the senses. (Parkinson's Mechanics, p. 1.)

which it moves, expressed by the number of feet described in a second ; then, $S = T \times V$.*

This is the fundamental equation of uniform motion, from which the other equations may be derived by the common rules of Algebra. For $S = T \times V$. $\therefore T = \frac{S}{V}$, and $V = \frac{S}{T}$; and if s=the space, t=the time, v=the velocity of any other body, expressed in the same manner; then the relation between S, T, V, and s, t, v, may be expressed by the following proportion ;

$$S : s :: T \times V : t \times v.\dagger$$

Adopting, therefore, the Algebraic notation for variable quantities,‡ we have $S \propto T \times V$; hence $T \propto \frac{S}{V}$, and $V \propto \frac{S}{T}$. If S be given, then $T \propto \frac{1}{V}$, and $V \propto \frac{1}{T}$.§

The Laws of Uniform Motion, therefore, are comprehended in the following THEOREMS, which are to be treasured up in the memory.

I. *The* SPACE *equals the product of the time into the velocity ;*‖ or (when different spaces are compared) the space *varies as* the product of the time into the velocity.

II. *The* TIME *equals the space divided by the velocity ;* or (when different times are compared) the time *varies as* the space divided by the velocity.

III. *The* VELOCITY *equals the space divided by the time ; or* (when different velocities are compared) the velocity *varies as* the space divided by the time.

IV. *When the space is given, the time varies inversely as the velocity.*

13. QUESTIONS ON UNIFORM MOTIONS.

1. *A ball was rolled on the ice with a velocity of* 30 *feet per second, and moved uniformly* 45 *seconds ; what* SPACE *did it describe?* Ans. 1350 feet.

* The young learner is apt to be puzzled with such abstract expressions as "*Space* equal to *time* multiplied into *velocity* ;" but it may be observed that by velocity is meant nothing more than the *space* passed over in *one second ;* which may evidently be so multiplied as to equal another space.

† Euc. V. 7. ‡ Day's Algebra, Art. 403, &c. § Do. 419.

‖ This is a concise mode of saying, The *number* expressing the space, equals the product of the *number* expressing the time into the *number* expressing the velocity.

2. *A steamboat moved steadily across a lake* 53 *miles wide, at the rate of* 16 *miles per hour; what* TIME *was occupied in crossing?* Ans. $3\frac{5}{16}$ hours.

3. *On the supposition that the earth describes an orbit of* 600 *millions of miles in* $365\frac{1}{4}$ *days, with what* VELOCITY *does it move per second?* Ans. 19 miles *nearly.*

4. *Three planets describe orbits which are to each other as* 15, 19, *and* 12, *in times which are as* 7, 3, *and* 5; *what are their* COMPARATIVE VELOCITIES? Ans. 225, 665 and 252.*

X.

14. *The* MOMENTUM *of a body is its quantity of motion, and is as the product of its quantity of matter and velocity.* The quantity of motion, or momentum, of each particle evidently depends on its velocity; and therefore the momentum of the whole must depend on the same particles multiplied into the common velocity. By velocity is understood the space moved over in a second. According to this definition, a body at rest cannot be said to have any momentum, but it is then said to have an amount of *inertia* corresponding to its mass. Inertia opposes the same resistance as momentum of similar amount.

Let M=the momentum of a body; Q=its quantity of matter or *weight* expressed in pounds; V=its velocity, expressed in feet; and let m=the momentum, q=the weight, v=the velocity, of any other body expressed in the same manner; then the relation between M, Q, V, and m, q, v, will be expressed by the following proportion; $M : m :: Q \times V : q \times v$.†

Hence $M \propto Q \times V$; $\therefore Q \propto \frac{M}{V}$ and $V \propto \frac{M}{Q}$.‡ If Q be given, $M \propto V$; if V be given, $M \propto Q$; and if $Q \propto \frac{1}{V}$, or $V \propto \frac{1}{Q}$, then M will be *given.*

The following THEOREMS therefore comprehend the doctrine of Momentum.

I. *The* MOMENTUM *equals the product of the quantity of matter into the velocity;* or (when different momenta are compared) the Momentum *varies as* the product of the quantity of matter into the velocity.

* Day's Algebra, 360, cor 1. † Euc. V. 7. ‡ Day's Algebra, Art. 403, &c.

II. *The* QUANTITY OF MATTER *equals the Momentum divided by the velocity;* or (when different masses are compared) the Quantity of Matter *varies as* the Momentum divided by the velocity.

III. *The* VELOCITY *equals the Momentum* divided by the quantity of matter; or (when different velocities are compared) the Velocity *varies as* the Momentum divided by the quantity of matter.

IV. *If the quantity of matter is given, the momentum is as the velocity.*

V. *If the velocity is given, the momentum varies as the quantity of matter.*

VI. *If two bodies move with velocities which are inversely as their quantities of matter, they have equal momenta.*

Thus, if a ship of 100 tons, sailing at the rate of 7 knots, meet another ship of 50 tons, sailing 14 knots per hour, they will encounter each other with equal momenta. This constitutes a fundamental principle in the mechanical action of bodies.

15. QUESTIONS ON MOMENTUM.

1. *A ship weighing* 336,000 *lbs. is dashed against the rocks in a storm, with a velocity of* 16 *miles per hour; with what momentum did she strike?* Ans. 7,884,800 lbs.

2. *On the supposition that Goliath of Gath presented an obstacle of* 350 *lbs., and that the stone hurled by David's sling weighed two ounces, with what velocity must it have been thrown to have prostrated the giant?*

Ans. It must have exceeded 2800 feet per second.*

3. *Wishing to know the velocity of a musket ball weighing* 1 *oz., I suspended, after the manner of a pendulum, a log of wood weighing* 53 *lbs. The ball on entering the log gave it a motion of* 2 *feet per second; what was the velocity of the ball?*

Ans. 1696 feet per second.

4. *If a comet moving at the rate of* 1,000,000 *miles per hour, were to meet the earth moving* 19 *miles per second, what ratio will the mass of the comet bear to that of the earth, supposing that they mutually destroy each other's motions?*

* The maximum velocity of cannon ball is usually reckoned 2000 feet per second.

Ans. 1 : 14,6; or the comet must have nearly $\frac{1}{15}$ as much matter as the earth.

5. *Two railway cars have their quantities of matter as* 7 *to* 3, *and their momenta as* 8 *to* 5; *what are their respective velocities?*

Ans. As 24 to 35, or nearly as 5 to 7.

16. FORCE *is any cause which moves or tends to move a body, or which changes or tends to change its motion.* (Art. 2.) Forces can for the most part be reduced to the three following classes, *attraction*, *repulsion*, and *animal strength*. Thus, the power of the waterfall can be traced to gravitation; that of steam to the repulsive energies of heat; and that of the horse and the ox to animal strength.

Forces are divided into two kinds, according to the manner in which they act. If a force acts *instantaneously*, and then ceases, it is called an impulsive force. A ball, suddenly put in motion by the hand or any instrument, along a horizontal plane, is an instance of the effect produced by an impulsive force. When a force acts *incessantly*, it is called an *accelerating* force, and is either constant or variable; *constant*, when the increments or decrements of velocity caused by it, in equal successive parts of time, are *equal;* and *variable*, when the increments or decrements of velocity thus produced are *unequal*. The force of gravity near the Earth's surface is an example of a *constant* force; for it causes equal increments or decrements of velocity in equal portions of time, not by *impulses*, but by *incessant* action. Gravity at different distances from the earth is a *variable* force, whose variation is estimated in the same manner as that of weight in article 8.

17. *Different constant forces generate velocities, which are as the product of the forces and times.*

Let T denote the time, and F the constant force; and conceive the time to be divided into exceedingly small equal portions; then since equal impulses, and of course equal velocities, are added to the moving body at each of these instants, the whole velocity acquired must be proportioned to that of each instant (which is the measure of F) multiplied by the number of instants; or $V \propto F \times T$.

One steam car was propelled by a constant force of 25 *lbs. for* 10 *minutes above what was sufficient to overcome all resistances,*

and another was driven by a similar force of 18 *lbs. for* 7 *minutes; what were their comparative velocities?*

Ans. As 250 to 126; or the first car had nearly twice the velocity of the second.

18. There are three great principles of motion called the Laws of Motion, derived from universal experience and observation, and of such extensive application as to comprehend all the phenomena of mechanics.

19. First Law.—*A body continues in the state in which it is, whether of rest or motion, until compelled by some external force to change its state.* That a body at rest will *continue* at rest, is a consequence immediately arising from the inertia of matter. That a body in motion *will continue to proceed uniformly along the right line in which it began to move, until it is acted upon by some external force,* is inferred from the fact, that any deviation from uniform rectilinear motion, in a moving body, is observed to be owing to *some external force;* and that such deviation is diminished in proportion as such external force is withdrawn; hence were it *entirely* withdrawn, we infer that the motion of the body would then become uniform, rectilinear and perpetual. We may see approximations to such a state in a ball rolled successively on the earth, on a floor, and on smooth ice. The most general impediments to motion are *friction, resistance of the air,* and *gravity.* But if a small wheel is put in motion round a horizontal axis, the effect of gravity is taken off, (since one side of the wheel gains as much in falling as the other loses in rising,) and no impediments remain but the resistance of the air and friction, the former of which may be removed by placing the apparatus in the vacuum of an air pump, and the latter may be greatly diminished by methods to be described hereafter. In proportion as these several impediments are removed, the wheel approximates to a motion which is uniform and continued. A pendulum has been constructed to move with so little resistance as, when barely set in motion with the finger, to continue to vibrate 24 hours.

20. Second Law.—*Motion, or change of motion, is proportioned to the force impressed, and is in the direction of that force.* It

has already been observed, that every change in the state of rest or motion in a body must be effected by the agency of some force; this Second Law asserts, that this change will in all cases be *proportional* to that force, and will be produced in *the direction in which the force acts.*

That motion or change of motion in a body will be proportional to the force which produces it, is also inferred from observation and experiment, as well as from the known connection between cause and effect. Thus a ball which moves with a double or triple velocity is found to generate in another, by impulse, a double or triple velocity. Two bodies meeting with equal quantities of motion mutually stop each other. Two forces, which, by acting similarly during equal times, produce equal velocities in some third body, are found by acting together during the same length of time, to produce a double velocity. If a new force is impressed upon a body in motion, *in the direction* in which it moves, its motion is increased proportionally to the new force impressed: if this force acts in a direction *contrary* to that in which the body moves, it is found to lose a proportional part of its motion: if the direction of this force is *oblique* to that of the moving body, it gives it a new direction compounded of both. A force which we know to act equally, produces equal increments of velocity in equal times.*

Hence it follows, that *the smallest force is capable of moving the largest body.*

With respect to the *direction* in which a body moves, it is evident that when it is under the direction of any given force, whether it be an impulsive one, or one that acts incessantly, the body can have no tendency whatever to deviate to the one side or to the other,† but must proceed along the right line in which the force acts.

21. THIRD LAW.—*When bodies act on each other, action and re-action are equal and in opposite directions.* The meaning of this law is, that when a body imparts motion in any direction, it loses an equal quantity of its own in the opposite direction—that when a body receives a blow, it gives to the striking body an equal blow—

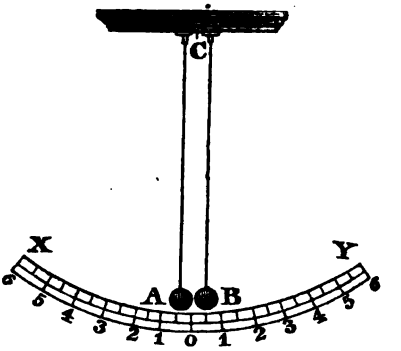
* Gregory's Mechanics, I. 9.

† According to the principle of the SUFFICIENT REASON, there being no cause why the body should deviate to one side of this line rather than the other; hence it will remain in it. (See Playfair's Outlines, I. 4.)

that when A presses on B, B returns to it an equal pressure—and that when it attracts or repels B, it receives from B the same influence in the opposite direction.

This law is brought to the test of experiment by means of the apparatus represented in Fig. 4, where A and B are two balls, of lead for example, suspended at C, by a flexible line, by which A may be drawn out towards X, and let fall upon B. The velocities gained or lost are indicated by the graduated arc XY; and it is found that when A falls on B, whatever motion A communicates to B, is communicated to A in the opposite direction; that is, the same amount is taken from A. Thus, if the two bodies are equal, and A falls on B at rest, they will, after the blow, move on together, with half the velocity of A, B having acquired, and A having lost an equal amount of motion. If A is greater than B, still it is found that the *momentum* gained by B (ascertained by multiplying its mass by the velocity) is precisely equal to the momentum lost by A; and if A meets B with a momentum greater than that of B, the latter will deprive A of a momentum equal to its own, and return along with A, both bodies having a momentum equal to the difference of their momenta previous to collision. It is a general Law in the Material World, that no body loses motion in any direction, without communicating an equal quantity to other bodies in the same direction; and conversely, that no body acquires motion in any direction, without diminishing the motion of other bodies by an equal quantity in that same direction.*

Fig. 4.



Now the moving force by which A communicates momentum to B, is called the *action* of A; and the tendency of B to diminish the momentum of A is called the *re-action* of B. Since, therefore, according to this meaning of the words action and re-action, the effect produced by the action of A is equal to the effect produced by the re-action of B, action and re-action are said to be *equal* during

* Playfair, Outlines, I. 8.

the impact of A upon B. That these effects are produced in "opposite directions," is evident from the very nature of the case.

This law applies not only to the *impact* of bodies, but to every case in which one body acts upon another. It holds good, not only when bodies come into actual contact, but when they act upon one another at any distance whatever. A body A for instance, is sustained by another body B, and both bodies remain at *rest;* if the pressure exerted by the two bodies were not *equal*, it is evident that some *motion* would ensue; which is contrary to the supposition. If motion does ensue, then the case becomes, in a great measure, analogous to that of *impact;* and the effects produced, estimated in a similar manner, are found to observe the same law.

The mutual *attractions* of bodies are also subject to this law. Thus, if two equal magnets, connected with two equal and similar pieces of cork, be made to float upon the surface of water, as soon as they come within the sphere of attraction, they are observed to move towards each other in a right line, with equal *velocities*, and consequently with equal *momenta;* and as the *resistance* which each body meets with from the fluid is evidently the same, we infer that their actions upon each other are *equal.**

22. These "fundamental principles of mechanics" rest on three different kinds of evidence:—

1. They are conformable to all *experience and observation.*
2. They are confirmed by various accurate *experiments.*
3. The *conclusions* deduced from them have always proved true in fact, without exception.†

OBSERVATION and EXPERIMENT, then, constitute the basis of the science of Mechanics. Observation is the close inspection, and attentive examination, of those phenomena, *which arise in the course of nature.*‡ Experiment is an *artificial trial* made for the purpose

* If two *unequal* magnets, placed upon pieces of cork similar to each other, and proportional to the respective magnets, were made to float in the same manner, they would approach each other with velocities inversely proportional to the quantities of matter moved, and consequently with *equal momenta;* but this experiment is liable to very great inaccuracy, from the different *resistances* which the bodies would meet with.

† See Gregory's Mechanics, I, 9. Atwood on Rectilinear Motion, p. 360.

‡ Leslie's Natural Philosophy, I, 2.

of learning the powers of nature, or the properties of substances. The most comprehensive results obtained by both these methods, so far as respects Mechanics, are expressed in the foregoing Laws of Motion. Upon these, therefore, the science of Mechanics is built. Algebra and Geometry, called in to the aid of these fundamental principles, lead to the discovery of new relations, and bring to light a great number of subordinate truths, of the highest degree of practical utility. *Granting the truth of the Laws of Motion,* as these subordinate truths are attained by principles purely scientific, (namely, those of Algebra and Geometry, and especially the latter,) they are attended with the evidence of demonstration; but since the conclusions can be no more certain than the premises, we can claim for the truths in Mechanics that degree of evidence only, which results from observation and experiment, applied in their greatest perfection.

23. Questions on the Principles of Motion.

1. *A bird of passage was observed to fly with a uniform velocity of* 19 *feet per second: over what* SPACE *would she pass in* 24 *hours?*

Ans. 310.909 miles.

2. *A lame man set out to travel round the world. He could walk but two miles an hour for seven hours out of the twenty four. Provided he could go forward, without impediment, on the circumference of a great circle of the globe,* (25,000 miles,) *what* TIME *would he require to complete the journey?* Ans. 4 years and $325\frac{5}{7}$ days.

3. *A wind blows uniformly from the equator to the pole,* (*say* 6000 miles,) *in* 12 *days: What is its* VELOCITY *per hour?*

Ans. $20\frac{5}{6}$ miles.

4. *How much weight would a rock that weighs ten tons* (22,400 *lbs.*) *at the level of the sea, lose if elevated to the top of a mountain five miles high?* Ans. 55.8952 lbs.*

5. *If the Earth were a hollow sphere, and if, through a hole bored through the center, a man were let down by a rope, would the force required to support him be increased or diminished as he descended*

* The weight would be ascertained, in this case, by the effect on a spring, (Art. 10,) and not by scales, since a counterpoise would sustain a loss of weight in the same degree with the body in question.

through the solid crust, and where would it become equal to nothing? (See Art. 9.)

6. *How much would a* 44 *pound shot weigh at the center of the earth; and how much at a point half way from the center to the surface?* (See Art. 10.)

7. *If a hole were bored through the center of the earth, and a stone were dropped into it, in what manner would the stone move in its way to the center and after it reached the center?* (See Arts. 9 and 10.)

8. *Suppose the battering-ram of Vespasian weighed* 5760 *lbs., and was found sufficient, when propelled with a certain velocity, to demolish the walls of Jerusalem; and suppose that a* 32 *pound cannon-ball, fired with a velocity of* 2000 *feet per second, is found capable of doing the same execution; what was the velocity of the battering-ram?* Ans. 11.11 feet per sec.

9. *Suppose a grain of light, moving at the rate of* 192,000 *miles per second, to impinge directly against a mass of ice moving at the rate of* 1.45 *feet per second; what weight of ice would the light stop?* Ans. 99877.832 lbs.,* or nearly $44\frac{1}{2}$ tons.

10. *If a ball of the same density with the earth,* $\frac{1}{10}$*th of a mile in diameter, were placed at the distance of* $\frac{1}{10}$*th of a mile above the earth; what space would the earth move through to meet it, the diameter of the earth being taken at* 8000 *miles?*

Ans. $\frac{1}{0.0000000000.00}$th inch, nearly.

11. *The quantity of matter in the sun being* 340,000 *times that of the earth, and their distances from each other being* 96,000,000 *miles; if the two bodies were abandoned to their mutual attraction, where would they meet?*

Ans. 282.35 miles from the center of the sun; or the sun would advance so far to meet the earth.†

12. *Two men are pulling a boat ashore by a rope, one at each end,* A *being in the boat and* B *on the shore: How will the time of bringing the boat ashore compare with the time in which* A *would pull it ashore alone, were the other end of the rope fixed to an immovable post?* (See Art. 21.)

* 1 lb. av. = 7000 grains.

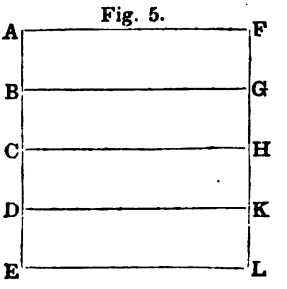
† In this problem it is supposed that all the matter of each body is collected in its center, (Art. 4,) and that their own semi-diameters present no obstacle to the approach of their centers.

CHAPTER II.

OF THE LAWS OF FALLING BODIES.

24. SINCE, when a body moves with a uniform velocity for a given time, the space described is in proportion to that time and velocity conjointly, (Art. 12,) therefore if one side of a right-angled parallelogram represents the *time* of a body's motion, and the other the uniform *velocity* with which it moves, the parallelogram itself (whose area is equal to the product of the two sides) will represent the *space* described in that time. Thus, let the line AE be divided into any number of equal parts in the points B, C, D, &c., and from those points draw the *equal* straight lines AF, BG, CH, &c., at right angles to AE, and complete the parallelogram AFLE; then if AB, BC, CD, &c., represent equal successive portions of *time*, and AF, BG, CH, &c., represent the uniform *velocity* with which a body moves, then will the parallelograms AG, BH, CK, &c., represent the *spaces* described in those equal portions of time, and the parallelogram AFLE the *whole space* described in the time represented by AE.*

Fig. 5.

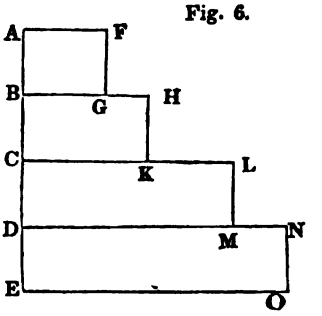


* Since the track described by a moving body is a *line*, how (it may be asked) can the space be properly represented by a *superficies?*

To avoid misconception on this subject, it will be useful for the young learner to recur to a few elementary principles. Quantity (it will be recollected) is any thing which can be *increased* or *diminished*, or which is capable of being *measured*. (Algebra, Art. 1.) Thus *time* is a quantity whose measure can be expressed in hours, minutes and seconds. *Velocity* is a quantity, being measured by the number of feet passed over in a second. But these two quantities (time and velocity) have no permanent representatives of their own, like numbers, which are represented by the digits, or like magnitudes, which are denoted by lines, surfaces, and solids. Hence, such quantities as times, velocities, and forces, are denoted by representatives *borrowed* from those of magnitude. In the case before us, the space described by a moving body is represented by a parallelogram, not because the space actually described has any *resemblance* to a parallelogram, but because a parallelogram has the *same relation* to the sides of which it is the product, as space has to the two quantities, time and velocity, of which it is the product. An *iden-*

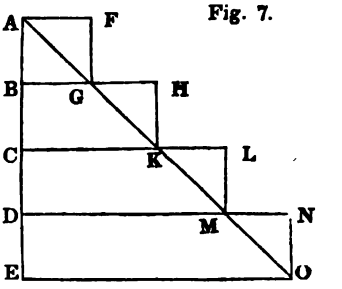
25. Suppose now that a body moves uniformly as before, during the equal successive portions of time represented by AB, BC, CD, &c., but at the end of each portion of time receives an *increase* of velocity; for instance, during the time AB let it move with a velocity represented by AF, during the time BC with a velocity represented by BH, &c.; complete the parallelograms AG, BK, CM, and DO, then the *space* described in the *time* AB will be represented by the parallelogram AG, in the *time* BC by the parallelogram BK, &c., and the *whole space* described in the time AE by the *irregular figure* AFOE.

Fig. 6.



26. Let us next suppose that a body receives *equal* increments of velocity at the end of each successive portion of time, so that during the *second* interval of time it moves with *twice* the velocity, during the *third* interval with *three* times the velocity, &c., then will CK=2BG, DM=3BG, &c. But AC=2AB, AD=3AB, &c.; ∴AC : CK :: AB : BG :: AD : DM, &c.: hence the figures ABG, ACK, ADM, &c., are similar triangles, and if AG, GK, KM, &c., be joined, AO will be a *straight line*, and the figure AFOE, which represents the *space* described in the *time* AE, will differ from the triangle AOE only by the sum of the triangles AFG, GHK, KLM, MNO, which are all equal to each other.

Fig. 7.

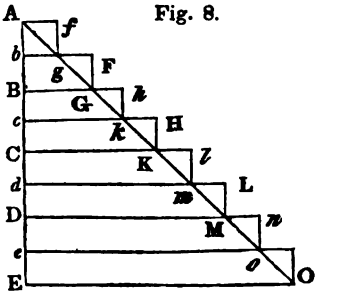


27. Now let the intervals of time and the corresponding increments of velocity be only *half* what they were in the former instance.

tity being thus established between the relations that subsist among magnitudes, and those that subsist among such quantities as have no representatives of their own, the representatives of magnitudes may be substituted to denote the relations of the other quantities; and thus a great number of new relations are frequently discovered to exist among those quantities, because they are known to exist among the magnitudes, as the lines, surfaces, &c., which are taken to represent them. It is thus that Geometry becomes a powerful auxiliary to Mechanics.

Bisect AB, BC, CD, &c., (Fig. 8,) in *b*, *c*, *d*, &c., and complete the parallelograms as before, then the figure which represents the space described in the time AE will differ from the triangle AOE by the sum of the small triangles A*fg*, *g*FG, G*hk*, &c., which is only *half* the sum of the triangles AFG, GHK, KLM, &c., in the preceding figure. By continually *halving* these intervals of time and the corresponding increments of velocity, the figure AFOE will approach to the form of the triangle AOE; and when they are diminished *ad infinitum* A*f*, (which represents the velocity with which the body begins to move) will be equal to 0; as will also the sum of the triangles A*fg*, *g*FG, &c.; the *space* therefore described by a body beginning to move from *rest*, by the *continued* action of a force which generates equal increments of velocity in equal times, will be accurately represented by the right-angled triangle AOE, one of whose sides (AE) represents the *time* of the body's motion, and the other (OE) the *last acquired velocity*.

Fig. 8.



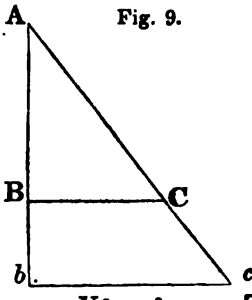
28. We have seen, (Art. 24,) that, in uniform motion, the time, velocity, and space, have the same relation to each other, as the sides and area of a right-angled parallelogram. Now if a moving body should retain all the velocity it has already acquired, and take on equal increments at equal successive instants, we see by the last article, that the whole space described would be represented by the sum of the parallelograms described successively. We see, moreover, that the smaller the time is taken, the nearer the whole space approaches to a right-angled triangle. But when a body is descending by the force of gravity, its velocity increases *continually*; the instant is reduced to nothing; and the little triangles which denote, in the other case, the difference between the figure described and that of a right-angled triangle, vanish, and leave the triangle as the proper representative of the space described.

The laws of variable motion, however, are more perfectly exhibited by means of the calculus, than they can be geometrically.*

29. *The spaces described by bodies falling from rest under the influence of gravity, are to each other as the squares of the times in which they are described, or as the squares of the last acquired velocities, or as the times and last acquired velocities conjointly.*†

For let S=the *space* described, V=the *velocity* acquired by a body falling from rest for the *time* T; s=space described, v=velocity acquired at any other period (t) of its fall; then, from what has already been demonstrated, if the ratio of T : t be represented by the lines AB, Ab, and the ratio of V : v by the lines BC, bc, drawn at right angles to them, the ratio of S : s will be represented by the triangles ABC, Abc. Now the triangle ABC : the triangle Abc :: AB^2 : Ab^2, or as BC^2 : bc^2, or as AB $\times$ BC : $Ab \times bc$.‡ Hence S : s :: T^2 : t^2, or as V^2 : v^2, or as T $\times$ V : $t \times v$.

Fig. 9.



As equal increments of velocity are generated in equal times, it is farther evident that the *velocity acquired varies as the time:* the same conclusion may also be deduced from the similar triangles ABC, Abc; for BC : bc :: AB : Ab, i. e. V : v :: T : t.

Since the spaces described are as the *squares* of the times; if a body falls from rest for *times* which are represented by the numbers 1, 2, 3, 4, 5, &c., the *spaces* described in those times will be as the square numbers, 1, 4, 9, 16, 25, &c.; and the spaces described in *equal successive portions of time* will be as the odd numbers 1, 3, 5, 7, 9, &c., as exhibited in the following table.

* See *Young*, Elements of Mechanics, art. 108. *Renwick*, p. 46.

† The demonstration applies to *any* uniformly accelerating force, as well as to gravity; and hence, although "Falling Bodies" are here under particular consideration, yet the proposition may be predicated of all bodies urged by uniformly accelerating or constant forces.

‡ For the right-angled triangles ABC, Abc, are to each other both as the squares of their homologous sides, (by Euc. 6, 19,) and in the ratio of the parallelograms of which they are respectively *halves*.

Times.	*Spaces described.*	*Spaces described in equal successive portions of time.*
1	1	In 1st portion of time - - - - 1
2	4	2d - - - - - 4 − 1 = 3
3	9	3d - - - - - 9 − 4 = 5
4	16	4th - - - - - 16 − 9 = 7
5	25	5th - - - - - 25 − 16 = 9
&c.	&c.	&c. - - - - - &c. = &c.

30. *If a body be projected perpendicularly upwards, with the velocity which it has acquired in falling from any height, it will rise to the point from which it fell, before it begins to descend again.*

As, in the *descent* of a body, the force of gravity generates equal increments in equal times, so, in its *ascent*, equal portions of velocity will be destroyed in equal times. The spaces described in equal successive parts of time, by a body thus ascending, reckoning from the beginning of its motion, will be the same as those stated in the foregoing table, but in an *inverted* order: thus, if the time be divided into four equal parts, then the spaces described in the *descent* of the body during these equal times are as the numbers 1, 3, 5, 7, but in its *ascent* they will be as 7, 5, 3, 1; i. e. the space described in the *first* portion of time, in its *ascent*, will be the same as that described in the *last* portion of time, in its *descent*, and so on, till the body arrives at its highest point.

31. *The space which a body describes from rest in any time, by the action of gravity, is* HALF *that which it would describe in the same time with the last acquired velocity continued uniformly.*

Let the triangle ABC represent the *space* described by gravity in the *time* AB, and BC the last acquired *velocity;* produce AB to D, making BD equal to AB, and complete the parallelogram BCDE; then, if a body moves for the time BD with the uniform velocity represented by BC, the space described in that time will be represented by the parallelogram BCDE (Art. 24); but the triangle ABC is *half* the parallelogram BCDE; hence the space described with the continually increasing velocity during the time AB is *half* that which would be described in the same time (BD) with the velocity (BC) continued uniformly.

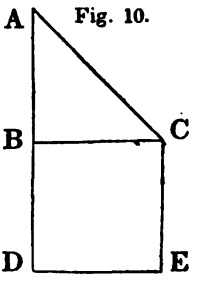


Fig. 10.

Since the space described by a body falling from rest, is *half* that which it would describe in the same time with its greatest velocity continued uniformly, and since a body projected perpendicularly upwards rises to the same height as that from which it must fall to acquire the velocity of projection, *the whole space described by a body projected perpendicularly upwards, is* HALF *that which it would describe in the same time with its first velocity continued uniformly.*

32. *The space described in any time by a body projected downwards with a given velocity, is equal to the space which would be described with that velocity continued uniformly for that time, together with the space through which a body would fall from rest by the action of gravity in the same time.*

Let AD represent the given *velocity* of projection, and AB the given *time*, and complete the right-angled parallelogram ABED; produce BE to C, and let EC represent the velocity generated by gravity in the time AB or DE, and join DC. Then, according to what has been said in Art. 24. a body moving under the influence of projection alone, with a uniform velocity represented by AD, would describe the parallelogram ABED in the time AB; and, by Art. 27. a body falling from a state of rest during the same time, so as to acquire the velocity represented by EC, would describe the triangle DEC; hence the figure ABCD truly represents the joint effects of both forces, or the whole space described.

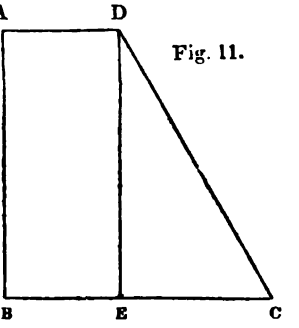


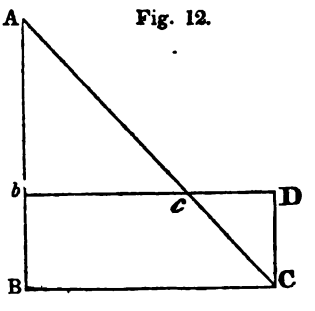
Fig. 11.

33. *The space described by a body* ASCENDING *for a given time is equal to the difference between the space which would be described by the body moving uniformly for that time with the velocity of projection, and the space through which a body would fall from rest by the action of gravity in the same time.*

Let BC (Fig. 12.) represent the given velocity of projection, and AB the time in which it must fall from rest to acquire that velocity;

draw BC at right angles to AB, and join AC, then the triangle ABC will represent the space through which the body must ascend to lose all its velocity. (Art. 30.) In AB take any point *b*, and complete the parallelogram BCD*b*; then will *bc* represent the velocity of the body at the end of the time B*b* of its ascent, and *c*D will represent the velocity destroyed by gravity in the same time. But the velocity *destroyed* by gravity in any time is equal to the velocity *generated* by gravity in the same time; (Art. 30.) hence the triangle CD*c* will represent the space through which a body would fall from rest in the time CD or B*b*. Now the figure BC*cb* represents the space through which the body would ascend in the time B*b*, and the parallelogram BCD*b* represents the space through which a body would move in the time B*b* with the velocity BC continued uniformly; but the figure BC*cb* is equal to the *difference* between the parallelogram BCD*b* and the triangle CD*c*.

Fig. 12.



34. The foregoing investigations show the *ratios* between the velocities, times, and spaces of falling bodies; but in estimating the actual motion of bodies descending or ascending by the force of gravity, it is necessary to have recourse to some *fixed standard* of measurement of *space* and *velocity*. Now it has been ascertained, by the most accurate experiments, that a body falling freely* from *rest* describes a space equal to $16\frac{1}{12}$ feet in the *first second* of its fall; and (Art. 31.) a body so falling would acquire a velocity which, if continued *uniformly*, would carry it over $32\frac{1}{6}$ feet (i. e. *twice the space*) in the same time. If, therefore, $m = 16\frac{1}{12}$ feet, m will express the *space* fallen through from rest in 1″, and $2m$ will express the *velocity per second* acquired in that time. Let S = the space described by the body in any other time T, and V the velocity acquired; then since the spaces are as the *squares of the times*, we have,

* All bodies descending or ascending near the surface of the earth, meet with more or less resistance from the air; so that, strictly speaking, a body can never be said to descend *freely* but in the exhausted receiver of an air-pump. It is in vacuo that a body describes $16\frac{1}{12}$ feet in a second; the conclusions, therefore, deduced in this section, will approximate to the truth only in those cases where the resistance of the air bears little or no proportion to the weight of the body.

$S : m :: T^2 : 1^2$, $\therefore S = mT^2$, and $T^2 = \frac{S}{m}$ or $T = \sqrt{\frac{S}{m}}$. The spaces are likewise as *the squares of the velocities* (Art. 29.); hence $S : m :: V^2 : \overline{2m}|^2 (4m^2)$, $\therefore S = \frac{mV^2}{4m^2} = \frac{V^2}{4m}$, and $V^2 = 4mS$ or $V = 2\sqrt{mS}$. Again, since *the time varies as the velocity*, (Art. 29.) we have $T : 1 :: V : 2m$, $\therefore V = 2mT$, and $T = \frac{V}{2m}$. Hence, of the three quantities S, T, V, any *one* being given, the *other two* may be found by the following RULES.

I. To find the SPACE fallen through in feet.—*Multiply the square of the time by* $16\frac{1}{12}$; or, *divide the square of the velocity by* $64\frac{1}{3}$.

II. To find the TIME of falling.—*Divide the velocity by* $32\frac{1}{6}$; or, *divide the space by* $16\frac{1}{12}$, *and take the square root of the quotient.*

III. To find the VELOCITY.—*Multiply the time by* $32\frac{1}{6}$; or, *multiply the space by* $16\frac{1}{12}$, *and double the square root of the product.*

1. *A body has been falling for* 6 *seconds; What space has it fallen through in that time, and what is the velocity which it has acquired?*

By Rule I. $S = mT^2 = 16\frac{1}{12} \times 36 = 579$ feet.

By Rule III. $V = 2mT = 32\frac{1}{6} \times 6 = 193$ feet per second.

2. *How far must a body fall to acquire a velocity of* 50 *feet in a second; and how long will it be in falling?*

Rule I. $S = 38.86$ feet.

Rule II. $T = 1.55$ seconds.

3. *A body fell from the top of a tower which was* 150 *feet high; How long was it in falling, and what velocity had it acquired when it got to the bottom?*

$T = 3.054$ seconds. $V = 98.237$ feet per second.

4. *A body was projected perpendicularly upwards with a velocity of* 100 *feet in a second; How far would it ascend before it began to return?*

By Art. 30, the height to which the body would ascend is equal to that through which a body must fall from rest to acquire the velocity of projection; here therefore $V = 100$, and $S = \frac{V^2}{4m} = \frac{10000}{4 \times 16\frac{1}{12}} = \frac{30000}{193} = 155.44$ feet.

5. *A body was observed to fall for* 3 *seconds, and afterwards to move uniformly for* 2 *seconds along the horizon with the velocity which it had acquired by its fall; what was the whole space described in its perpendicular and horizontal motion?*

The space described in its fall $=mT^2=16\frac{1}{12}\times 9=144\frac{3}{4}$ feet. The velocity acquired $=2mT=32\frac{1}{6}\times 3=96\frac{1}{2}$ feet per second; and as it moved along the horizon for 2 seconds with this velocity, it must, in that time have described 193 feet; hence the whole space described from the beginning of its fall to the end of its horizontal motion is $144\frac{3}{4}+193$, or $337\frac{3}{4}$ feet.

6. *A cannon ball fired perpendicularly upwards, was gone* 10 *seconds, when it returned to the same place. How high did it rise, and what was the velocity of projection?*

Ans. Height $402\frac{1}{12}$ feet. Velocity of projection $160\frac{1}{3}$ feet per second.

35. Since the spaces described in equal successive parts of time (by Art 29.) are as the odd numbers 1, 3, 5, 7, 9, &c., and since the space described by a body falling from rest is in the *first* second m feet, the space described in *successive* seconds will be m, $3m$, $5m$, $7m$, $9m$, &c. feet.

1. *A body had been falling for* 5 *seconds; compare the spaces described in the third and fifth seconds of its fall.*

Ans. The space described in the *third* second $=80\frac{5}{12}$ feet; the space described in the *fifth* second $=144\frac{3}{4}$ feet.

2. *A body has fallen through* 579 *feet; what was the space described by it in the last second?*

Ans. It will be found that the body has been falling 6 seconds, therefore, the space described in the last second is $176\frac{1}{12}$ feet.

36. The method adopted in the last example for finding the space described by a body in the *last* second of its fall, is only applicable when the time consists of a *determinate* number of seconds; but it is not difficult to investigate a *general* expression for the space described in the last n seconds, whatever be the value of T. For the *space* described in T seconds $=mT^2$; and the space described in $T-n$ seconds $=m\times\overline{T-n}|^2=mT^2-2mnT+mn^2$; hence the space de-

scribed in the last n seconds $= mT^2 - \overline{mT^2 - 2mnT + mn^2} = 2mnT - mn^2 = m.\overline{2nT - n^2}$; if $n=1$, then the space described in the *last* second $= m.\overline{2T-1}$; which expression will lead to the same results as the method practiced in example 1, and is likewise applicable in cases where the time does not consist of any even number of seconds. For example, let the time of falling be $6\frac{1}{2}$ seconds; then the space described in the last second, namely, from $5\frac{1}{2}$ to $6\frac{1}{2}$ seconds, will be $16\frac{1}{12} \times \overline{13-1} = 193$ feet.

If it were required to find the space described in the second *immediately previous* to the last n seconds, we have,

Space described in the last n seconds $= m.\overline{2Tn - n^2}$ (A).

Ditto in last $\overline{n+1}$ seconds $= m.\overline{2Tn+1-n+1|^2}$ (B).

Subtract (A) from (B), then the space described in the second immediately previous to the last n seconds $= m.\overline{2T-2n-1}$.

1. *What was the space described in the last 2 seconds by a body which had fallen from the top of a tower 300 feet high?*

Ans. The whole time is found to be 4.32 seconds; therefore, the space fallen through in the last 2 seconds is 213.58 feet.

2. *A body has been falling for $9\frac{1}{2}$ seconds; what was the space described in the last second but 4 of its fall?* Ans. $160\frac{1}{8}$ feet.

37. *To find the space described in a given time by a body projected upwards or downwards with a given velocity.* Let V= the given velocity with which a body is projected *downwards*, and T= the time of its motion; then the space described in the time T with the uniform velocity V will be equal to $T \times V$, and the space through which a body would fall by gravity in the same time is mT^2; hence, from what was shown in Art. 32, the space described in the time T by a body projected *downwards* with the velocity V is equal to $T \times V + mT^2$; and applying the same process of reasoning to Art. 33, the space through which a body would ascend in the time T, if projected *upwards* with a given velocity V, will be equal to $T \times V - mT^2$.

1. *A body is projected downwards with a velocity of 30 feet in a second; how far will it fall in 4 seconds?* Ans. $377\frac{1}{2}$ feet.

2. *A body is projected upwards with a velocity of 120 feet in a second? how far will it rise in 3 seconds?* Ans. $215\frac{1}{2}$ feet.

38. Miscellaneous Examples.

1. *With what velocity must a body be projected downwards from the height* (a), *that it may describe it in* T *seconds?*

Let x = the velocity required; then the space described by a body projected downwards with velocity (x) in the time (T) is $Tx+mT^2$; hence $Tx+mT^2=a$, $\therefore x=\frac{a-mT^2}{T}$. For instance, let $a=150$, and $T=2$, then the velocity with which a body must be projected downwards from the top of a tower whose height is 150 feet, so that it may arrrive at the bottom in *two* seconds $=42\frac{1}{2}$ feet per second.

2. *With what velocity must a body be projected from the top of a tower* 300 *feet high, to reach the ground in* 4 *seconds?*

Ans. $10\frac{1}{2}$ feet per sec.

3. *The space described by a heavy body in the* 4*th second of its fall was to the space described in the last second except* 4, *as* 1 *to* 3; *what was the whole space described by the body?*

The space described in the 4th second $=7m$; the space described in the last second but 4 $(=m.\overline{2T-2n-1})=m.\overline{2T-9}$, where T = the *whole time* of falling; hence from the question $7m : m.\overline{2T-9} :: 1 : 3$, $\therefore 2T-9=21$, or $T=15$; the whole space described therefore $(=mT^2)=16\frac{1}{16}\times 225=3618\frac{3}{4}$ feet.

4. *Suppose at the same instant that a body begins to fall from rest from the point* (D), *another body is projected upwards with a velocity which would carry it to* A; *it is required to find the point where they would meet.*

Let C be the point where the bodies would meet; and let $AB=a$, $BD=b$, $DC=x$; then will $AD=a-b$, $AC=a-b+x$. Now the time of *descending* through $DC=\sqrt{\frac{x}{m}}$; and the time of *ascending* through BC (= time down AB − time down AC) $=\sqrt{\frac{a}{m}}-\sqrt{\frac{a-b+x}{m}}$, but the time *down* DC must be equal to the time *up* BC; hence we have

Fig. 13.



$$\sqrt{\frac{x}{m}}=\sqrt{\frac{a}{m}}-\sqrt{\frac{a-b+x}{m}},$$

$$\text{or } \sqrt{x}=\sqrt{a}-\sqrt{a-b+x};$$

$$\therefore \sqrt{a-b+x}=\sqrt{a}-\sqrt{x},$$

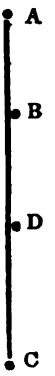
and $a-b+x=a+x-2\sqrt{ax}$;

$\therefore 2\sqrt{ax}=b$, or $4ax=b^2$, and $x=\frac{b^2}{4a}$.

5. *Suppose a body to have fallen from* A *to* B, (Fig. 14.) *when another body begins to fall from rest at* D; *how far will the latter body fall before it is overtaken by the former?*

Let C be the point where one body overtakes the other, and let $AB=a$, $BD=b$, $DC=x$; then $AC=a+b+x$. Now time down $DC=\sqrt{\frac{x}{m}}$, and time down BC = time down AC − time down $AB=\sqrt{\frac{a+b+x}{m}}-\sqrt{\frac{a}{m}}$; but at the moment when the lower body is overtaken,

Fig. 14.



Time down DC = time down BC,

or $\sqrt{\frac{x}{m}}=\sqrt{\frac{a+b+x}{m}}-\sqrt{\frac{a}{m}}$;

$\therefore \sqrt{x}+\sqrt{a}=\sqrt{a+b+x}$,

and $x+a+2\sqrt{ax}=a+b+x$,

or $2\sqrt{ax}=b$, $\therefore x=\frac{b^2}{4a}$.

39. Questions on Falling Bodies.

1. *From a black cloud a flash of lightning was observed, and* 15 *seconds afterwards it began to rain: on the supposition that the rain began to fall on the instant of the flash, what was the height of the cloud?** Ans. 3618.75 feet.

2. *A meteoric stone fell upon a projecting stick of timber, with a momentum which, from the motion given to the stick, was estimated at* 18435 *pounds. It occupied in falling* 10 *seconds: from what height did it fall, and what was the weight of the stone?*

Ans. Height, 1608⅓ feet; Weight, 57.31 lbs.

3. *A man fell into a pit* 500 *feet deep: how long was he in falling, and what velocity did he acquire?*

Ans. T = 5.57 seconds; V = 179.17 feet per sec.

* No allowance is here made for the resistance of the air, which, in fact, greatly retards the descent of drops of rain.

4. *Wishing to ascertain the difference in the depths of two wells, I dropped a pebble into one of them, and heard it strike the water in* 6 *seconds; and then into the other, and heard it strike in* 7 *seconds: what was the difference of their depths?* Ans. $209\frac{1}{7}$ feet.

5. *An archer wishing to know the height of a tower, found that an arrow sent to the top of it, occupied* 8 *seconds in going and returning: What was the height of the tower?* Ans. $257\frac{1}{2}$ feet.

6. *In what time would a man fall from a balloon three miles high, and what velocity would he acquire?*

Ans. T=31.38 seconds, or about half a minute.

V=1009.39 feet per second, or about half the maximum velocity of a cannon ball.*

7. *A body having fallen for* $3\frac{1}{2}$ *seconds, was afterwards observed to move along the horizon (with the velocity which it had acquired in its descent) for* $2\frac{1}{2}$ *seconds; What was the whole space described by the body from the beginning of its fall?*

Ans. $478\frac{1}{2}$ feet, very nearly.

8. *Through what space would the äeronaut (in question* 6.*) fall during the last second?* Ans. 993.3 feet.

* In this problem we have an example of the immense velocity which bodies falling towards the earth from a great height finally acquire, being, in the case supposed, more than eleven miles per minute. To form an idea of the whole progress of a very distant body falling towards the earth, we must conceive of it as at first moving with extreme slowness, and as accelerated by very small increments of velocity. For, although in terrestrial mechanics, gravity is considered as a *constant* force, producing *uniform* acceleration, yet it must be remembered that it is, in fact, a *variable* force, diminishing as the square of the distance increases; and hence, that the acceleration which it produces in a given time, is not only much less at remote distances from the earth than at its surface, but that the *rate* of acceleration itself is constantly increasing as we approach the earth. The principles, therefore, which serve for estimating the time, velocity, and space, of bodies falling near the earth, as in the foregoing examples, do not answer for bodies falling from great distances. The laws governing these are investigated by means of the Calculus. Since the accelerating force diminishes so rapidly as the distance from the earth increases, there is a limit to the velocity which a body can acquire by gravity. If it falls from an infinite distance, it can acquire only a velocity of about *seven miles* per second; and half of this is gained within 1354 miles of the earth. Were a body projected from the earth with a velocity of 7 miles per second, it would never return. (See Vince's Fluxions, Sec. VIII, Prop. XL. 5. Young's Elements of Mechanics, Art. 116.)

9. *A body has fallen from the top of a tower* 340 *feet high; What was the space described by it in the last three seconds?*

Ans. 298.957 feet.

10. *Suppose a body be projected downwards with a velocity of* 18 *feet in a second; How far will it fall in* 15 *seconds?*

Ans. $3888\frac{3}{4}$ feet.

11. *A body is projected upwards with a velocity of* 65 *feet in a second; How far will it rise in* 2 *seconds?* Ans. $65\frac{1}{3}$ feet.

12. *With what velocity must a stone be projected into a well* 450 *feet deep, that it may arrive at the bottom in* 4 *seconds?*

Ans. $V=48\frac{1}{2}$ feet in a second.

13. *Upon a steeple* 160 *feet high, is a spire of* 50 *feet; at the same instant that a stone was let fall from the top of the steeple, another was projected perpendicularly upwards from the bottom of it, with a velocity sufficient to carry it to the top of the spire; At what point will these stones meet?*

Ans. 96.96 feet from the top of the steeple.

14. *Upon the top of a tower* 200 *feet high, is placed a flag-staff of* 26 *feet; a bullet is let fall from the top of this flag-staff; and at the instant of its passing the bottom of it, a stone is let fall from a window* 44 *feet from the top of the tower; At what distance from the bottom of the tower will the bullet overtake the stone?*

Ans. 137.385 feet.

CHAPTER III.

OF THE COMPOSITION AND RESOLUTION OF MOTION.

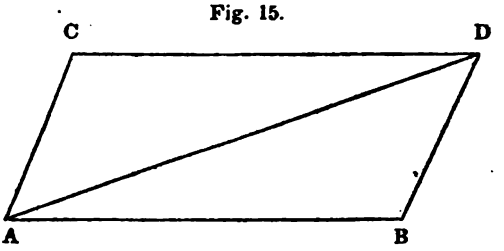
40. In the two preceding Chapters, we have considered the motion produced in bodies by the action of only a *single* force. We now proceed to show the manner in which a body would move when acted upon at the same time by *several* forces. Let us first consider the case of a body acted upon by *two* forces.

41. *Two impulses, which, when communicated separately to a body would make it describe the adjacent sides of a parallelogram in a given time, will, when they are communicated at the same in-*

stant, cause it to describe the diagonal in that time; and the motion in the diagonal will be uniform.

Suppose a body placed at A to be acted upon by two forces, one of which would cause it to move uniformly over the line AB, and

Fig. 15.



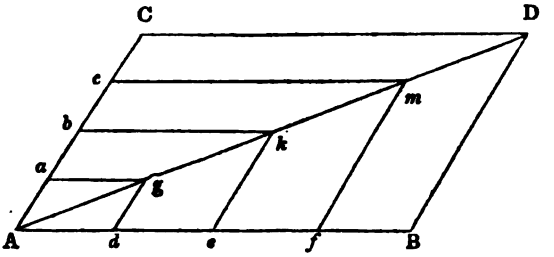
the other over the line AC in the same given time, then complete the parallelogram ACDB; and if both forces act at the same instant upon the body, it will, by their *joint* action, move uniformly over the diagonal AD in the same time that it would have described either of the sides AB or AC by the forces acting separately. For if we attentively consider the manner in which these forces act, it is evident that the force which acts in the direction AB, can have no tendency whatever to prevent the access of the body towards the line CD, which is parallel to AB (2d Law of Motion, Art. 20.)* When both forces act together, therefore, it will, by the action of the force in the direction AC, arrive at the line CD (but in a *different point* of that line) in the same time as if the force in the direction AB had not acted; for the same reason, the force in the direction AC will have no tendency to prevent the access of the body towards BD, which is parallel to AC; it will arrive therefore at the line BD in the same time as if the force in the direction AC had not acted. Hence the body will arrive at the lines CD and BD at the same

* "When any force is exerted upon a body already in motion, the motion which the force would produce upon a body at rest, is compounded with the previous motion in such a way, that *both produce their full effects parallel to their own directions.*" (Whewell, 228.) That any force impressed upon a body, already moving under the influence of different forces, has its full effect, either in producing or in destroying motion, is evident from the fact that a given force is found to have *the same effect* upon bodies in different parts of the earth, although, in consequence of the diurnal motion of the earth, bodies apparently at rest are moving with various velocities in different latitudes." (Ib. 231.)

instant of time, and consequently will be found at their common intersection D; and as the body, after it leaves the point A, is acted upon by no external force, it must, by the *first law of motion*, have described the diagonal AD with a uniform motion.

42. As this motion of a body in the *diagonal* of a parallelogram *by the joint action of two forces which (acting separately) would have caused it to describe the two sides*, is a fundamental theorem with respect to the *composition of motion*, let us consider it in another point of view. Let the lines AC, AB be divided

Fig. 16.



into the same number of small equal parts A*a*, *ab*, *bc*, &c.; A*d*, *de*, *ef*, &c., which will be to each other as the whole lines AC, AB, i. e. A*a* : A*d* :: AC : AB; *ab* : *de* :: AC : AB; &c., and consequently (Alg. Art. 388.) A*a*+*ab* (A*b*) : A*d*+*de* (A*e*) :: AC : AB; &c.; if therefore the parallelograms A*dga*, A*ekb*, &c., be completed, then (by Euc. 6. 26.) the points *g*, *k*, *m*, &c., will all fall in the diagonal AD. Now since AC, AB are described uniformly in the same time, the proportional parts A*a*, A*d*; A*b*, A*e*, &c. will be described uniformly in the same time. From what has already been demonstrated therefore, at the end of those different parts of time the body will be brought to the point *g*, *k*, *m*, &c. by the united action of the forces which would have separately made it move over A*a*, A*d*; A*b*, A*e*, &c. Let the number of parts into which AC, AB are divided be indefinite, then the number of points *g*, *k*, *m*, &c. will be indefinite, and the lines A*g*, *gk*, *km*, &c. will be indefinitely small; the body therefore will *begin* to move in the line A*g*, and, being found at the end of each successive instant of

time in the line AD, it must have moved over that line with the uniform velocity with which it set off.*

43. *If a body be acted upon by two forces, one of which would cause it to move uniformly over one side of a triangle, and the other over another side of the triangle, in the same time, then by the joint action of those forces it will be made to describe the third side, in the same time that it would have described either of the sides by the forces acting separately.*

Thus if a body be acted upon by two forces, one of which would cause it to move uniformly over the side AC, (Fig. 15.) and the other over the side CD, of the *triangle* ACD, (or over a line parallel and equal to CD,) then, by the joint action of those forces, it would be made to describe the third side AD in the same time that it would have described either of the sides AC, CD, by the forces acting separately. For if the parallelogram ACDB be completed, then, since AB is equal and parallel to CD, a force acting in the direction AB would make a body describe AB in the same time as that in which it would describe CD; but by what has already been proved, if two forces act upon a body, by one of which it would be made to describe AC, and by the other AB, in the same time, it would by the joint action of these forces be made to describe the diagonal AD, which is the *third side* of the triangle ACD.

44. Let us next suppose a body placed at A, and acted upon by *three* forces, by *one* of which it would be made to describe AB, by *another* AC, and by the *third* AD, uniformly in the same time; complete the parallelogram ABEC, and join AE: complete also the parallelogram AEFD, and join AF; then AF will be the line over

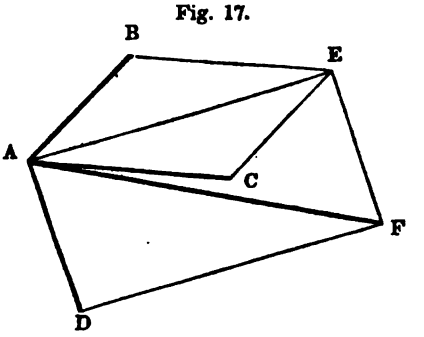
* The body will also describe the diagonal AD, when acted upon by uniformly *accelerating* forces. For, let Ae, Ad, (Fig. 16.) be two spaces described in the times T, t, by one uniformly accelerating force, and Ab, Aa, be two spaces described by another similar force, then,

$$Ae : Ad :: T^2 : t^2$$
$$Ab : Aa :: T^2 : t^2 \therefore$$
$$Ae : Ad :: Ab : Aa;$$

consequently, the points g, k, which denote the positions of the body at the end of each successive instant, are all in the same straight line. (Euc. VI. 26.)

which the body will move uniformly by the joint action of those forces, in the same time in which it would have described AB, AC, or AD, by either of the forces acting separately. For by Art. 41. a body acted upon by two forces in directions AB, AC would be made to describe the diagonal AE; a body placed at A therefore, and acted upon by *three* forces in directions AB, AC, AD, is under the same circumstances as if it was acted upon by two forces one of which would make it describe AE, and the other AD in the same time; but the line over which a body would move uniformly by the action of two forces in the directions AE, AD is the diagonal AF; AF therefore is the line over which it would move uniformly by the joint action of the three forces in the directions AB, AC, AD.

Fig. 17.



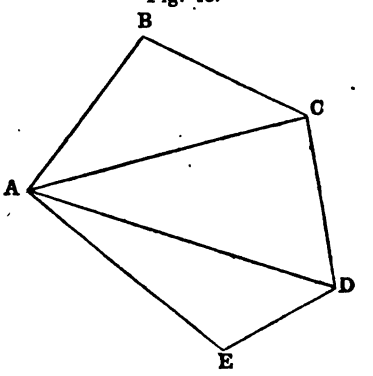
Since BE is equal and parallel to AC, (Fig. 17.) and EF equal and parallel to AD, it follows, (for the same reason as in Art. 43.) that if a body be acted upon by three forces, each of which acting separately would make it describe in *succession*, the three sides AB, BE, EF of the figure ABEF, (or lines parallel and equal to them,) taken in the order of the letters, A, B, E, F, it would by the *joint action* of those forces be made to describe the fourth side AF in the same time that it would have described those sides respectively when the forces act separately. And since the same mode of reasoning applies to a polygon of any number of sides, we have *in general*, the following theorem.

45. *If a body be impelled by any number of forces, which acting separately, would, in a given time, make it describe all the sides of a polygon except the last side; when all these forces act at the same instant, it will be made to describe the remaining side in the same given time.*

Thus, if a body be impelled by any number of forces which acting *separately*, would, in a given time, make it describe each of the

sides AB, BC, CD, DE of the *polygon* ABCDE; when all those forces act *at the same instant*, it will be made to describe the remaining side AE in the same given time.

Fig. 18.



46. *If all the sides of a polygon except the last represent the quantity and direction of several forces, acting at the same instant upon a body, the remaining side will represent the quantity and direction of a force equivalent to them all.*

A force is said to be EQUIVALENT to any number of forces, when it will, singly, produce the same effect that the others produce jointly in any given time. The single force is frequently called the RESULTANT, and the forces that produce it are called the COMPONENTS.

By the second law of motion, *the space described is proportional to the force impressed;* in all these cases, therefore, the *spaces* respectively described by the body will represent the *quantity and direction* of the forces by which it is impelled. Thus (see Fig. 15.) if the *quantity and direction of two forces* be represented by the two sides AB, AC of the parallelogram ACDB, the *diagonal* AD will represent the quantity and direction of a force *equivalent to them both;* or if the *two sides* AC, CD of the triangle ACD represent the quantity and direction of two forces acting at the same time upon a body, the third side AD will represent a force equivalent to them both. With respect also to the forces by which a body is made to describe the sides AB, BC, CD, DE, of the *polygon* ABCDE (Fig. 18.); if AB, BC, CD, DE represent the quantity and direc-

tion of several forces acting at the same instant upon a body, the remaining side AE will represent the quantity and direction of a force equivalent to them all.*

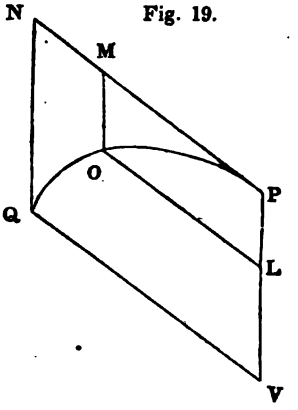
47. Since the lines which represent the proportion of the forces in these different figures are described in the same time, and since the velocity of a body is proportional to the space described in a given time, these lines will also represent the proportion of the velocities with which they are respectively described. Thus (Fig. 15.) the velocity with which the diagonal AD is described : velocity with which either of the sides AC or AB is described : : AD : AC or AB; and in the case of the polygon in Art. 45, the velocity with which the side AE is described : velocity with which either of the sides AB, BC, CD, or DE is described : : AE : AB, BC, CD or DE.

48. Hitherto the forces have been supposed to be such as by their *separate* action would produce *uniform* velocities; in which case, a body by their *joint* action will be made to describe a straight line with a uniform velocity. But if two forces act upon a body, by one of which it would be made to describe a straight line with a *uniform* velocity, and by the other with a *variable* velocity, then the body, by the united action of those forces, will neither describe a *straight* line, nor will it move with a *uniform velocity;* but will describe with a variable velocity some *curve* line, the form of which must be determined from the particular nature of the two forces which act separately upon the body. Let us take the case of a body projected obliquely at the earth's surface, on the supposition that it meets with no resistance in its passage through the air. Conceive a body to be projected from the point P, in the direction PN, with such velocity as would carry it uniformly over the line PN in the same time that it would descend by the force of gravity through the space PV. Complete the parallelogram PNVQ; then for the same reason as in Art. 41, the body at the end of that given time would be found in the

* It will be remarked by the learner, that several of these forces acting in opposite directions, partly destroy one another, so that AE represents merely the *resultant*, or what remains after all these mutual actions.

point Q; having described *not the diagonal* PQ, but some *curve line* POQ.* In PN take any point M, and let T, t represent the times of describing PN, PM respectively; make PL equal to the space through which a body would fall by gravity in the time (t), and complete the parallelogram PMOL; then O will be the place of the body at the end of the time (t); and in the same manner the other

Fig. 19.



points of the curve POQ might be determined. Now since PN is described with a uniform velocity,

$$PN : PM :: T : t \therefore$$
$$PN^2 : PM^2 :: T^2 : t^2. \text{ But, (Art. 29.)}$$
$$NQ : MO :: T^2 : t^2 \therefore$$
$$NQ : MO :: PN^2 : PM^2.$$

Hence the curve is such, that $MO \propto PM^2$, which is a well known property of the parabola.† The curve POQ, therefore, is a parabola whose diameter is PV, *ordinate* QV, and whose *parameter* to the point P is $\frac{QV^2}{PV}$‡ $= \frac{PN^2}{PV}$.

* For the body in descending in the direction of PV, recedes from the line PN very slowly at first, but faster and faster as it proceeds; and since the rate of acceleration is augmented *continually*, the body must be constantly drawn farther and farther from the direction of PN. But a constant change of direction, implies that the path is a curve line.

† Bridge's Conic Sec. Art. 27.

‡ By Cor. to Prop. 8 of *parabola*, parameter $\times PV = QV^2$, $\therefore$ parameter $= \frac{QV^2}{PV}$. (Bridge's Conic Sections.)

49. Questions on the Composition of Motion.

1. *A body is acted upon at the same time by two forces which are to one another as a : b, and their directions are inclined to each other in the given angle* A; *What is the magnitude of the resultant?*

Let AC : AB (Fig. 15.) represent the ratio of $a : b$, and let BAC be equal to the given angle A. Complete the parallelogram ABDC, then AD will represent the resultant. Since CD=AB, AC : CD :: $a : b$; and since CD is parallel to AB, the angles BAC, ACD=180°, $\therefore$ ACD=180° − ∠A; hence the problem is reduced to finding trigonometrically the third side AD of the triangle ACD, in which are given the two sides AC, CD, and the included angle ACD.

Let AC : AB :: 2 : 3, and ∠A=60°, $\therefore$ ACD=120°; then* 5 : 1 :: tan. 30° : tan. $(\frac{1}{2}\text{CAD}-\frac{1}{2}\text{CDA}) = \frac{1}{5}\text{tan. } 30°$; $\therefore$† log. tan. $\frac{1}{2}\text{CAD}-\frac{1}{2}\text{CDA}$=log. tan. 30° − log. 5=9.0624694=log. tan. 6° 35′; hence CAD=36° 35′, and CDA=23° 25′, and as sin. 23° 25′ (CDA) : sin. ACD (120°) :: AC (2) : AD=4.36, i. e. if two forces which are to each other as 2 : 3 act upon a body at an angle of 60°, the resultant will be proportional to 4.36.

2. *From an island in the Straits of Sunda, we sailed S. E. b S.* (33° 45′) *at the rate of* 6 *miles an hour; and being carried by a current, which was running towards the S. W.* (*making an angle with the meridian of* 64° 12½′,) *at the end of four hours, we came to anchor on the coast of Java, and found the said island bearing due north. Required the length of the line actually described by the ship, and the velocity of the current?* Ans. S=26.4 mile.,
V=3.7024 miles per hour.

3. *A sloop is bound from the main land of Africa to an island bearing W. b N.* (78° 45′) *distant* 76 *miles, a current setting N. N. W.* (22° 30′) 3 *miles an hour: What is the* COURSE *to arrive at the island in the shortest time, supposing the sloop to sail at the rate of* 6 *knots per hour; and what* TIME *will she take?*

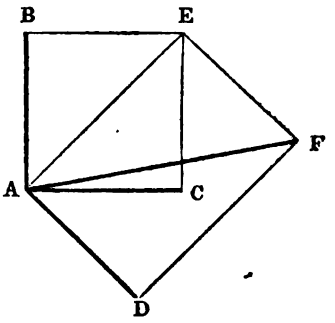
Ans. Course 76° 41′ 4″ S.—Time 10h. 40m. 7 sec.

* Day's Trigonom. Art. 144. † Ib. Art. 41.

50. We may likewise find the magnitude of the force compounded of any number of forces, whose quantities and directions are represented by the sides of the given polygon ABCDE, (Fig. 18.) For since AB, BC and ABC are given, AC and BCA may be found; but ACD=BCD−BCA, ∴AC, CD and ACD are known, from which AD and ADE may be determined; and in the triangle ADE we have AD, DE, and ADE, ∴AE is known.

Three men are pulling at a boat with equal forces and in the same plane. A *pulls at right angles to* B, *and* B *at an angle of* 45 *degrees with* C. *In what* DIRECTION *will the boat move, and what is the ratio of the* RESULTANT *to the sum of the individual forces?*

Fig. 20.



Let the point of application be at A, (Fig. 20.) and let AB, AC, and AD, represent respectively the magnitude and direction of A, B, and C. Then it may be shown that AF is the resultant, which makes an angle with AB of 80° 16′, and bears to the sum of A, B and C, the ratio of $\sqrt{3}$ to 3.

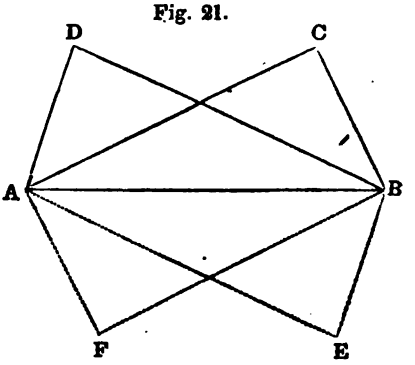
RESOLUTION OF MOTION.

51. *A given force may be resolved into an unlimited number of others, acting in all possible directions.*

Let AB represent the quantity and direction of some given force; draw any lines AD, AC, and join DB, CB; complete also the parallelograms ADBE, ACBF. Since AB is the diagonal of two parallelograms whose adjacent sides are respectively AD, AE, and

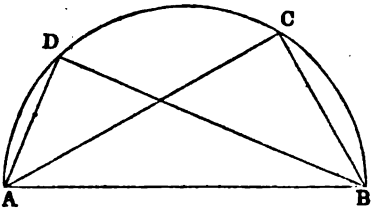
AC, AF, it may (by Art. 41.) be considered as the resultant of two forces whose quantities and directions are represented either by AD, AE, or AC, AF, i. e. by AD, DB or AC, CB. The forces represented by AD, DB, or AC, CB, may also be resolved into other pairs of forces, and so on without end.

Fig. 21.



52. Sometimes, however, by the conditions of the problem, the resolved forces are required to make a *given* angle, or to be in a certain ratio, with each other. The method of solving cases of this kind, may be illustrated by a few examples.

Fig. 22.

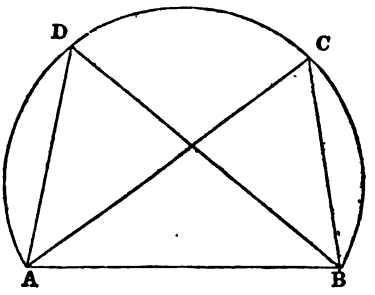


First, let a given force AB, (Fig. 22.) be resolved into pairs of forces which shall always act at *right angles* to each other. Upon AB describe a semicircle, ABC, and from the extremities of the base draw straight lines to meet in any point of the circumference. The side of the triangles thus formed, will severally contain right angles.*

* Euc. III. 31.

Secondly, let the resolved forces be required to make with each other *any* given angle. Upon AB, (Fig. 23.) describe the segment of a circle, ABC, containing an angle, the supplement of the given angle,* and draw straight lines from the extremities of the base to any point in the circumference.

Fig. 23.



Thirdly, let the *sum* of the resolved forces be required to be equal to a given quantity. Let MN (Fig. 24.) be equal to the sum of the forces required and AB be the given force; and upon MN, as the transverse, and with the points A and B (equally distant from M, N,) as foci, describe the ellipse MCN. From A and B draw straight lines to any point in the ellipse, and the sides of the several triangles will form the pairs of forces required.†

Fig. 24.

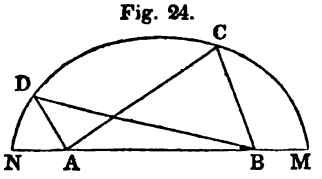
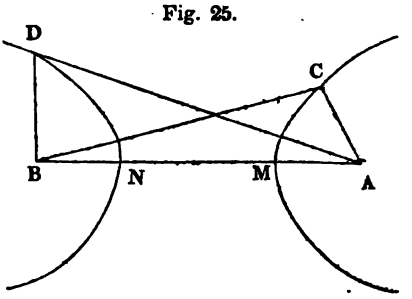


Fig. 25.

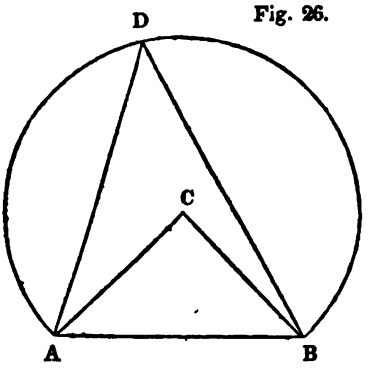


Fourthly, in like manner, pairs of forces whose *difference* shall be always equal to the same constant quantity, may be found by making A and B the foci of an hyperbola, as in Fig. 25., and drawing straight lines from these points to the curve.‡

* Euc. III. 33. † Bridge's Conic Sect. Art. 8. ‡ Conic Sect. Art. 9.

1. *A given force* (a) *is required to be resolved into different pairs of forces which shall act at an angle of* 135° *to each other; What is the radius of the circle whose segment shall contain pairs of the resolved forces?*

Fig. 26.



Let AB$=a$, and upon AB describe the segment of a circle which shall contain an angle ADB of 45°: then the radius AC$=\frac{a}{\sqrt{2}}$.

2. *To determine the radius of the circle when* AB (Fig. 26.) *is required to be resolved into pairs of forces acting at any given angle whose supplement is* A.

Let AB$=a$, ADB$=$A; find the center C, and join CA, CB; then BC or radius $=\frac{a \times \cos.A}{\sin.2A}$.

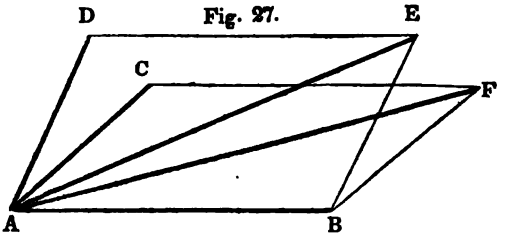
Let $a=10$, A$=40°$ } then radius $=7.7786$.

RELATIVE MAGNITUDE OF FORCES BEFORE AND AFTER THEIR COMPOSITION OR RESOLUTION.

53. The most obvious consideration with respect to the composition of motion is, that, if two equal forces act upon a body in contrary directions, they will destroy each other's effects, and the body thus acted upon will remain at rest; or if any two forces act upon a body in the same straight line, then the effect (or, in other words, the motion) produced will be proportional to the sum or difference of those forces according as they act in the same or in opposite directions. But if these forces act obliquely to each other, then the re-

sulting force will be some intermediate quantity between that sum and difference, the magnitude of which will increase according as the angle of inclination between the directions of these forces is diminished. For it is evident that the smaller the angle of inclination between two forces is, the more nearly will they conspire together, and consequently the whole effect produced will be greater; on the contrary, as the angle of inclination increases, the two forces will more strongly oppose each other, their whole effect therefore will keep diminishing.

Fig. 27.



This latter conclusion may also be drawn from the *geometrical* representation of the forces. Let two forces be represented by AB, AD, or by AB, AC, of which AC=AD; let the angle DAB be greater than the angle CAB, and complete the parallelograms DABE, CABF; then since DAB is greater than CAB, its supplement ABE must be less than ABF the supplement of CAB; hence in the triangles ABE, ABF, we have AB, BE equal to AB, BF, and the angle ABF greater than ABE, ∴ *AF is greater than AE. Let CAB=0, then AF=AC+CF=sum of the forces; let CAB =180°, then AF=CF − AC=difference of the forces; in all other cases, AF is of some intermediate magnitude between AC+CF and CF−AC, and keeps increasing as the angle CAB is diminished.

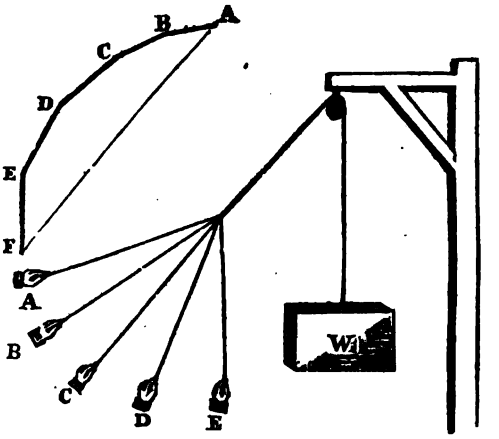
54. In the composition of forces which act obliquely on each other, some force is actually lost; for the sum of the forces before they are compounded together is represented by the two sides AD, DE of a triangle, and after composition by the third side AE. The contrary happens with respect to the resolution of forces; for the two resolved forces being represented by the two sides of a triangle of which the given force is the third, the absolute quantity of the resolved forces must be greater than that of the given force.

* Euc. I. 24.

Five sailors raise a weight by means of five separate ropes, in the same plane, connected with the main rope that is fastened to the weight in the manner represented in figure 28. B *pulls at an angle with* A *of* 20°; C *with* B, *at* 19°; D *with* C, *at* 21° 30′; *and* E *with* D, *at* 25°. A, B, *and* C, *pull with equal forces, and* D *and* E *with forces, one half greater: Required the magnitude and direction of the resultant, and the loss of force occasioned by the forces acting partly against each other?*

Let the sides of the polygon (Fig. 28.) represent the several forces in magnitude and direction, then AF will be the resultant.

Fig. 28.



The angle at B=160°; at C, 161°; at D, 158° 30′; at E, 155°. Hence,

1. The resultant makes an angle with AB=46° 33′ 10″.

2. Its value is 5.1957, (that of all the components being 6,) and it falls between C and D.

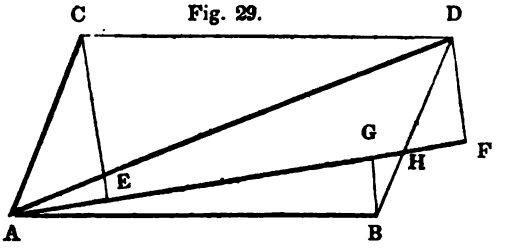
3. The loss of force is .1341, or about $\frac{1}{7}\frac{3}{4}$ of the whole.

But the effect of forces to produce motion in *given directions* is not changed by their composition or resolution. For let the two forces AC, AB (Fig. 29.) and their resultant AD, be resolved into the several forces AE, EC; AG, GB; and AF, FD; whereof AE, AG, AF, are in the direction of some line AF given in position* and

* Algebra, Art. 17.

EC, GB, DF, perpendicular to it; then by similar triangles GHB, HDF, we have,

Fig. 29.



GH : HF::BH : HD, and GH+HF : HF::BH+HD : HD,
∴GF : HF::BD : HD, or GF : BD (=AC) ::HF : HD.
But by similar triangles AEC, HFD, HF : HD ::AE : AC.
Hence GF : AC::AE : AC, ∴GF=AE, and AF=AG+AE;
i. e. that part of the compound force AD which acts in the direction AF is equal to such parts or the resolved forces as act in the same direction.

Again, by similar triangles, BG : DF::BH : HD;
∴BG+DF : DF::BD or AC : HD,
or BG+DF : AC::DF : HD::CE : AC.

Hence BG+DF=CE, ∴DF=CE−BG ; but since CE, BG, represent forces acting in opposite directions to each other, their effect in the direction CE perpendicular to AF will be represented by their difference CE−BG; ∴ since DF=CE−BG, the part of the compound force which is perpendicular to AF is equivalent to such parts of the resolved forces AC, AB as are perpendicular to AF.

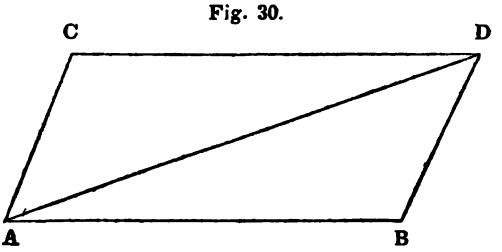
NATURE OF THE FORCES, WHICH, BY THEIR JOINT ACTION, WILL KEEP A BODY AT REST.

55. *A body acted upon at the same time by three forces, represented in quantity and direction by the three sides of a triangle taken in order, (or by lines parallel to these,) will remain at rest.*

Since AD (Fig. 30.) is equivalent to AB and AC, a body placed at A and urged by AB and AC in one direction, and by DA in the opposite direction, it would remain at rest. But these three forces correspond in magnitude and direction with the three sides of the triangle ACD.

56. *If a body be kept at rest by three forces, those three forces will be represented by the three sides of a triangle formed by lines drawn in their respective directions.*

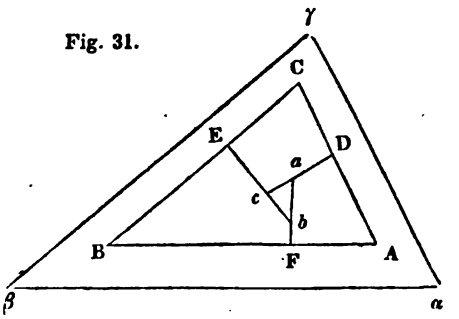
Fig. 30.



For suppose a body be kept at rest by three forces, and that AC, CD (Fig. 30.) represent the quantities and directions of two of those forces, then the compound force arising from those two forces will be represented by the line AD; a third force, therefore, represented in quantity and direction by the line DA, equal and opposite to AD, must exactly counterbalance AD and keep it at rest. Whenever, therefore, a body is kept at rest by three forces, if a triangle be drawn, whose sides are respectively in the direction of those forces, those sides will represent the quantity and direction of the several forces thus acting upon the body.

57. *The proportion of the three forces which keep a body at rest will be represented by the three sides of any triangle, drawn parallel or perpendicular to the sides of the triangle which are in the directions of the forces.*

Fig. 31.



For, let the triangle ABC be that whose sides are drawn in the direction of the three forces, then the triangle $\alpha\beta\gamma$ (whose sides are

parallel to AB, BC, CA) and the triangle *abc* (whose sides are *perpendicular* to AB, BC, CA) being each of them similar* to the triangle ABC, must have their sides $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$, or *ab*, *bc*, *ca*, respectively proportional to the three sides AB, BC, CA, which represent the quantity and direction of the forces acting upon a body.

58. *Any one of the three forces which keep a body at rest, is as the sine of the angle included between the other two.*

For, (Fig. 31.)AB : BC::sin.BCA : sin.BAC; BC : CA::sin. BAC : sin. ABC; and CA : AB::sin. ABC : sin. BCA. Conversely, *if a body be acted upon by three forces, each of which varies as the sine of the angle included between the directions of the other two, it will remain at rest,* since the sines are as the sides opposite to them, and when the forces are proportional to these sides, the body will remain at rest by Art. 55.

59. *A body will be kept at rest if it be acted upon by any number of forces which are represented in quantity and direction by the sides of a polygon taken in order.*

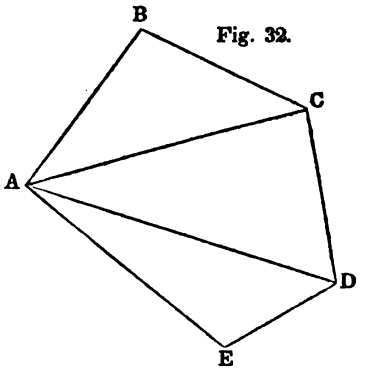


Fig. 32.

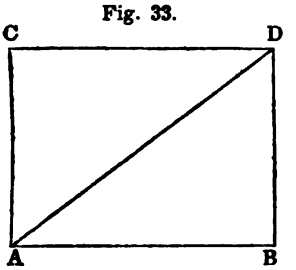
For, let a body be acted upon by any number of forces represented by the sides AB, BC, CD, DE of the polygon ABCDE;

* Since the sides of the triangle $\alpha\beta\gamma$ are respectively parallel to the sides of the triangle ABC, it is evident that the angles $\alpha\beta\gamma$, $\beta\gamma\alpha$, $\gamma\alpha\beta$, are respectively equal to ABC, BCA, CAB. With respect to the triangle *abc*; since the angles at D, E, F, are *right*-angles, we have $DaF+DAF=180°$, also $DaF+bac=180°$, $\therefore$ $DAF=bac$; and in the same manner it appears that $EBF=abc$, and $ECD=acb$. (See Legendre's Geometry, III. 209.)

then (by Art. 44.) these forces compounded together will be represented in quantity and direction by the remaining side AE; if, therefore, at the same time that the body is acted upon by the forces AB, BC, CD, DE, it is also acted upon by another force represented in quantity and direction by EA, (equal and opposite to AE,) it will remain at rest. The converse of this proposition may also be established by the same mode of reasoning as that made use of in Art. 56.) viz. *If a body be kept at rest by any number of forces, those forces will be represented, in quantity and direction, by the sides of a polygon formed by the intersection of lines drawn in the direction in which they respectively act.*

1. *A body is acted upon by two forces a and b, which are at right angles to each other; it is required to find the magnitude and direction of a third force, which shall keep the body at rest.*

Fig. 33.



Let $AC=a$, $AB=b$, $CAB=90°$ } Complete the parallelogram ABDC, and join AD; then the two forces acting upon the body may be represented by AC, CD; consequently DA (the third side of the triangle ACD) will represent the force which shall keep the body at rest. (Art. 55.) Now $DA=\sqrt{AC^2+CD^2}=\sqrt{a^2+b^2}$ for the magnitude of the force; and sine of CAD : sine of CDA (=DAB) :: CD : CA :: b : a; $\therefore$ the direction DA of the third force divides CAB into two angles, whose sines are to each other as a : b.

2. *A body, acted upon by two forces, is kept at rest by a third force (a), whose direction divides the angle contained between the directions of the two former into the given angles* A *and* B; *What is the magnitude of those two forces?*

Let AC, AB be the two forces (Fig. 30.); complete the parallelogram ABDC, then DA (=a) is the third force; let CAD=A, DAB=B, then $ACD=180°-\overline{A+B}$. Now

$$AC : AD(a) :: \sin.CDA \text{ or } DAB(B) : \sin.ACD\,(180^\circ - \overline{A+B},)$$
$$\& AB : AD(a) :: \sin.ADB \text{ or } CAD(A) : \sin.ACD\,(180^\circ - \overline{A+B}.)$$
$$\text{Hence } AC = \frac{\sin.\ B \times a}{\sin.\ 180^\circ - \overline{A+B}}, \text{ and } AB = \frac{\sin.\ A \times a}{\sin.\ 180^\circ - \overline{A+B}}.$$

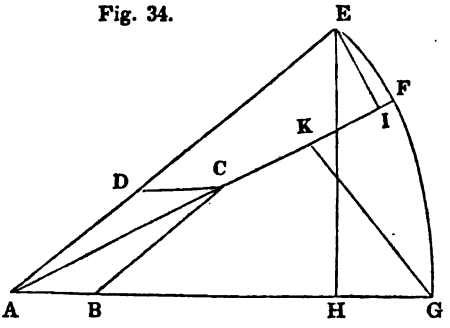
COMPOSITION AND RESOLUTION OF PARALLEL FORCES.

60. It appears by Art. 58. that any one of the three forces which keep a body at rest is proportional to the sine of the angle included between the directions of the two others. An important consequence of this truth is, that, of three forces that keep a body at rest, the two components and the resultant may severally be represented by the sine of the angle included between the directions of the two others; viz. the resultant by the sine of the angle comprehended between the directions of the two components, and each of the components by the sine of the angle comprehended between the resultant and the other component. Hence are derived the principles of the composition and resolution of parallel forces.

Thus, let a body at A be kept at rest by three forces represented in quantity and direction by the sides of the triangle ADC, AD and DC being the components and AC the resultant. Complete the parallelogram ABCD; produce the lines AB, AC, AD, indefinitely, and with any radius describe the arc EFG and from the points E, G, draw the sines EH, EI, GK.

From what has been said it will be seen, that the force AB may be represented by the sine EI, the force AD by GK, and the force

Fig. 34.



AC by EH. Suppose now that the lines AE, AF, AG, approach towards parallelism; then it is evident that the center A will con-

tinually recede from the arc EFG, and that that arc will continually approach towards a straight line while the sines will approach towards a coincidence with it, until finally, when the lines AE, AF, AG, become parallel, the sine EH will cross the parallels at right angles, and the sines EI and GK will form parts of the same straight line with EH. Hence, when the two forces AB and AD become parallel, their resultant AC forms another parallel with them both; and since the resultant is represented by the sine EH, which, when these lines become parallel, equals the two sines EI and GK, we hence derive the following THEOREMS.

I. *The resultant of two parallel forces is in a direction constituting another parallel, and is equal to their sum.*

II. *If a straight line be drawn perpendicular to the directions of these three forces, (viz. the two components and their resultant) each of the components will be represented by the part of the perpendicular contained between the directions of the two others.*

61. We have thus far considered the two parallel forces as acting the same way; when they are directed towards opposite parts, the investigation is the same with that in the last article, and the conclusion the same, except that the resultant is equal to the difference of the two components. A great number of parallel forces may be compounded into a single force equivalent to them all by proceeding as in Art. 44; that is, by first finding the resultant of two forces, and a new resultant for that resultant and one of the remaining forces, and so on to the last; and any single force may be resolved into any number of parallel forces by a method the reverse of this.

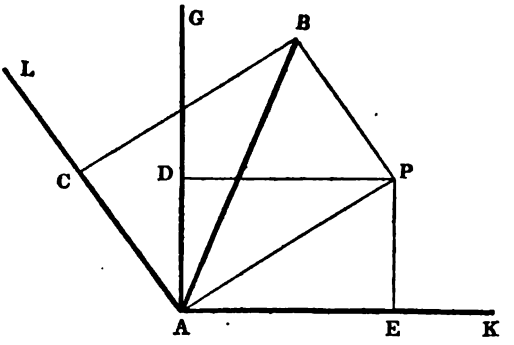
COMPOSITION AND RESOLUTION OF FORCES ACTING IN DIFFERENT PLANES.

62. In estimating the effects produced by the composition and resolution of forces, we have hitherto considered them as acting in the same plane; we proceed now to the solution of the problem, by means of which we are enabled to determine the motion of a body resulting from the operation of any number of forces acting in different planes.

All the forces which can possibly act upon a body, may be resolved into equivalent forces acting in the direction of THREE STRAIGHT LINES OR AXES, ***at right angles to each other.***

Let AK, AG be two straight lines drawn at right angles to each other in the same plane, and let AL be drawn at right angles to that plane, and consequently, at right angles to each of the lines AK, AG.* Suppose AB to represent the quantity and direction of a force acting upon a body at A; let fall the perpendicular BP upon the plane passing through AK, AG; join AP, and complete the parallelogram APBC. From P draw PD parallel to AK, and PE parallel to AG. Since AB is the diagonal of the parallelogram APBC, the force represented by AB is resolved into two others AC, AP equivalent to it; and since AP is the diagonal of the

Fig. 35.



parallelogram ADPE, the two AD, AE are equivalent to AP. Hence the given force AB is resolved into three others AC, AD, AE, in the direction of the three straight lines AL, AG, AK, which are at right angles to each other, and issue from the point A.

Produce LA to *l*, GA to *g*, and KA to *k*; so that the three lines (or axes) L*l*, G*g*, K*k*, shall cut each other at right angles in the point A; then it is evident that the directions of all the forces, which can possibly act upon a body at the point A, will fall within one or other of the eight solid angles formed by the intersection of three

* Euc. 2 Sup. Def. I.

planes cutting each other at right angles, and passing through the axes L*l*, G*g*; L*l*, K*k*; G*g*, K*k*, respectively; and from what has just been shewn, each of those forces may be resolved into *three* others, in the directions,*

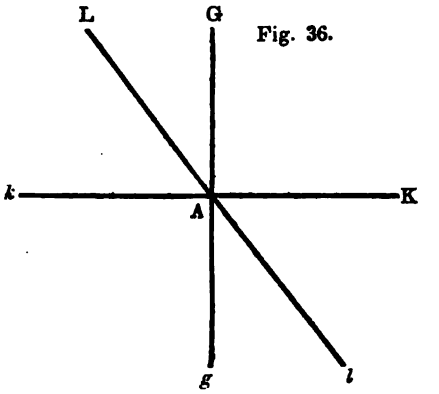


Fig. 36.

AL, AG, AK;
AL, AG, A*k*;
AL, A*g*, A*k*;
AL, A*g*, AK;
or A*l*, AG, AK;
A*l*, AG, A*k*;
A*l*, A*g*, A*k*;
A*l*, A*g*, AK;

according to the solid angle in which it is included. Thus, then, all the forces which can possibly act upon a body at the point A, may be resolved into others acting along the three axes L*l*, G*g*, K*k*; for the forces acting in the directions AL, A*l*; AG, A*g*; AK, A*k*, respectively, are merely forces acting in *opposite* directions in the same straight line.

Cor. If the sum of the opposite forces in the direction of each axis be equal to one another, the body will be at rest.

63. Miscellaneous Questions on the Composition and Resolution of Motion.

1. *Three men, equal in strength, undertake to pull down the steeple of an ancient church. They fasten three ropes to a ring near the top, and standing at equal distances from the circular base of the steeple, they pull at equal angles of* 30° *to each other. The ropes severally make an angle of* 40° *with the perpendicular axis of the steeple. Now if a single force of* 500 *lbs. were applied at right angles at the same point, it would be just sufficient to overturn the steeple: Required the force actually exerted by each man?*

Ans. 284.717 lbs.

* It will be observed that the first four angles lie *above* and the last four *below* the plane that passes through AK and AG.

2. *A body at the equator moves by the diurnal revolution of the earth about* 1000 *miles, and in lat.* 40°, *about* 766 *miles per hour. Now were a wind to blow from the equator, commencing with a course directly north, and blowing with a uniform velocity of* 60 *miles per hour, in what* DIRECTION *would it blow when it reached the latitude of* 40°, *supposing it still to retain the easterly motion it had in common with other bodies at the equator?*

Ans. N. 75° 37′ E.

3. *If a man were taken up at the latitude of* 40° *and at the same instant set down at the equator, in what* DIRECTION *and with what* VELOCITY *would he move on the equator?*

Ans. He would move directly westward, at the rate of 234 miles per hour.

4. *A ferry boat crosses a river* $\frac{3}{4}$ *of a mile broad in* 45 *minutes, the current running all the way at the rate of* 3 *miles an hour: At what* ANGLE *with the direct course must the boat head up the stream in order to move perpendicularly across?* Ans. 71° 34′.

5. *The same things being given, in what* RATIO *is the force required to move the boat* INCREASED, *in consequence of the current?*

Ans. It is increased 3.162 times.

6. *I shot an eagle that was flying directly over my head. On account of its inertia, it retained some motion in a horizontal direction, and therefore fell, at the end of* 4 *seconds,* 60 *feet from the place where I stood: Required the nature of the* CURVE *which the bird described in its fall?*

Ans. The curve is a PARABOLA, of which the *equation** is $P \times 257\frac{1}{8} = 3600$; and consequently the *parameter* (P) = 13.99. (See Art. 48.)

* Bridge's Conic Sec. Prop. 8.—Cor.

CHAPTER IV.

OF THE CENTER OF GRAVITY.

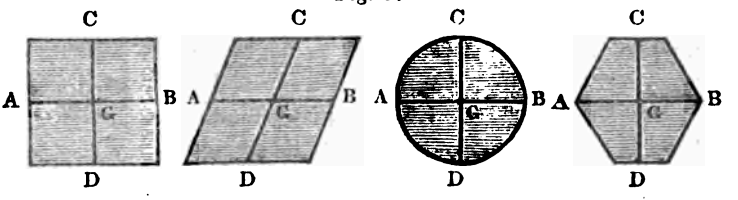
64. *The center of gravity of a body is that point, about which, if supported, all the parts of a body (acted upon only by the force of gravity) balance each other in any position.*

In order to ascertain this point, it will be necessary to resolve a body into its constituent parts; and then to find two lines, about each of which (if supported) these parts will balance each other in all positions: the common intersection of those two lines will be the center of gravity required. In bodies of a regular form and uniform texture this is very easily effected, but the difficulty increases as the nature or shape of the body becomes more complex. We shall begin with showing the method of finding the center of gravity of a body, or of a *system* of bodies in a few familiar instances.

METHOD OF FINDING THE CENTER OF GRAVITY OF A BODY OR SYSTEM OF BODIES.

65. *In regular plane figures, such as squares, parallelograms, circles, polygons inscribed in circles, &c. the center of gravity is the same as the center of the figures.*

Fig. 37.

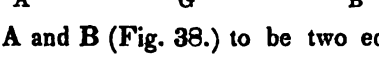


Let the annexed figures represent thin laminæ of matter of a uniform density, and let them be divided into two equal parts by the straight lines AB, CD. Conceive now each of those laminæ to be resolved into lines of particles equal and parallel to AB, there will then be the same quantity of matter similarly disposed on each side of AB; if, therefore, AB be supported, the parts ACB, ADB will

balance themselves about it; the center of gravity will consequently be in the line AB. For the same reason, because all lines drawn parallel to AB are bisected by CD, the center of gravity will also be in the line CD; it must therefore be in their common intersection G. In the same manner it might be shown, that the center of gravity of regular solids, such as the cube, parallelopiped, cylinder, sphere, &c. is the same with the center of magnitude. For each of these solids might be divided into two equal and similar parts by planes passing through it in three different directions; the intersection of two of these planes would be a right line, and the intersection of that line with the third plane would be the center of gravity of the solid. Let the three first figures in the preceding page represent respectively the section of a cube, parallelopiped and sphere, cut through their middle; then may the line CD represent the intersection of a plane at right angles to ACBD, and AB the intersection of a third plane cutting these solids in a similar manner; the point G will therefore be the center of gravity. In the cylinder, it is evident that the center of gravity will be in the point which bisects its axis.

66. *When a body is supported by a prop placed under its center of gravity, the pressure is the same whether the whole quantity of matter is uniformly diffused through the space occupied by the body, or whether it is all concentrated in that center of gravity.*

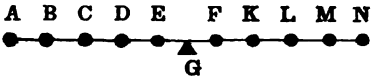
Fig. 38.



Suppose now A and B (Fig. 38.) to be two equal particles of matter connected together by the inflexible rod AB void of gravity; bisect AB in G, then G will be the common center of gravity of A and B; for it is evident, that if G be supported, the two particles will balance themselves about it. The pressure upon G will be equal to the weight of the particles A and B, and this pressure does not at all depend upon the length of the line AB; it will therefore be the same whether the particles be placed at A and B, or a particle equal to A+B be placed at G. The same may be said with respect to the particles A, B, C, D, &c. (in Fig. 39.) which are disposed uniformly along the inflexible rod AN void of gravity;

viz. that the pressure of A and N is the same as if A+N were placed at G; of B and M the same as if B+M were placed at G; and that the whole pressure of the particles A, B, C, D, &c. is the same as if A+B+C+D+E+F, &c. were placed at G.*

Fig. 39.



This reasoning might be extended to the lines of particles composing the laminæ in Art. 65. for the particles A, B, C, D, &c. (Fig. 39.) may be increased in number till they become contiguous to each other, and the effect is the same whether we consider them as connected together by an inflexible rod void of gravity, or as actually united together by the power of cohesion. Supposing CD therefore to be supported, (see figures in Art. 65.) the pressure upon it will be the same as if all the particles contained in the lines parallel to AB were incumbent upon it; and supposing the point G only to be supported, the pressure will be the same as if the particles thus collected in CD were incumbent upon it; the pressure of the laminæ ACBD upon the center of gravity is therefore the same as if all the matter contained in it were incumbent upon G. The same mode of demonstration might be applied to the laminæ composing the regular solid bodies in Art. 65.

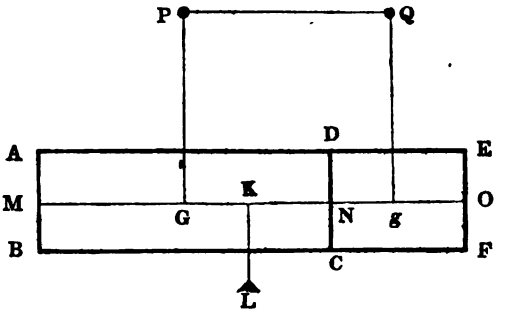
67. *Two weights or pressures acting at the extremities of an inflexible rod void of gravity, will be in equilibrio about a given point, when their distances from that point are to each other inversely as those weights or pressures.*

Let ABCD, CDEF, (Fig. 40.) represent the sections of two cylinders of uniform density and of the same diameter, whose axes are MN, NO; bisect MN in G and NO in g, then will G, g, be their centers of gravity. Let ABCD be suspended from the hook P, by the string PG attached to its center of gravity; and let CDEF be suspended in the same manner from the hook Q, by the string Qg; let them also be so placed that their ends may be contiguous to each other. Then will ABCD, balance itself about G, and

* We shall come to the same conclusion by considering A and B, &c. as parallel forces, and G as their resultant; then, by Art. 60., G will be equal to their sum.

CDEF about g, so that the two axes NM, NO (after the cylinders are suspended) will lie in the same straight line; and the pressure upon G, g, will be the same as if the whole weights of the cylinders were collected respectively in those points. Suppose now the

Fig. 40.



two ends which are contiguous to each other, to be firmly cemented together, so that the two cylinders should become one mass; this will not at all affect the pressures upon G, g, but will merely serve to connect those two points together in such a manner that the pressures upon them may be considered as acting at the extremities of an inflexible rod Gg void of gravity. Bisect the axis MO of the whole cylinder ABFE in the point K, and K will be its center of gravity; let the prop KL be placed under K, and let the two strings PG, Qg, by which it is suspended, be removed, and the cylinder will then balance itself about the point K; or, in other words, the two pressures acting at G, g, will be in equilibrio about that point. It only remains, therefore, to find the relation of KG to Kg; now $MK=\frac{1}{2}MO$, and $MG=\frac{1}{2}MN$, $\therefore MK-MG$ (or KG) $=\frac{1}{2}(MO-MN)$ $=\frac{1}{2}NO$; again, $OK=\frac{1}{2}MO$, and $Og=\frac{1}{2}NO$, $\therefore OK-Og$ (or Kg) $=\frac{1}{2}(MO-NO)=\frac{1}{2}MN$; hence KG : K$g$:: $\frac{1}{2}$NO : $\frac{1}{2}$MN :: NO : MN :: cylinder CDEF : cylinder ABCD. But the pressure upon G(P) is equivalent to the weight of the cylinder ABCD, and the pressure upon g (p) to that of the cylinder CDEF; $\therefore$ KG : Kg :: p : P or P : p :: Kg : KG.

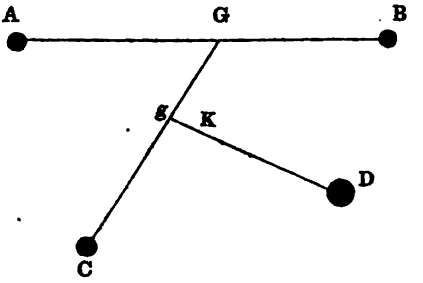
68. This furnishes us with the method of finding the common center of gravity of any number of bodies whatever, connected together by inflexible rods void of gravity.

Let A, B, C, D, &c. be the bodies, and let the centers of gravity of A and B be connected together by the inflexible line AB;

take A : B::BG : AG,*

or,† A+B : B::BG+AG(AB) : AG, then will the bodies A

Fig. 41.



and B balance themselves about G, (Art. 67.) and consequently G will be their common center of gravity (by Art. 64.); and the three first terms of the above proportion being known, the distance of G from A is thus found.

Next, let the center of gravity of C be connected with G by the inflexible line CG, then for the reason assigned in Art. 66. the pressure upon G will be the same as if a body equal to A+B were placed at G; take, therefore,

A+B : C::C*g* : G*g*,

or, A+B+C : C::C*g*+G*g*(CG) : G*g*, then *g* will be the center of gravity of A+B and C, and consequently the common center of gravity of the three bodies A, B, C.

Again let the center of gravity of D be connected with *g* by the inflexible line D*g*, then the pressure upon G will be the same as if A+B and C were placed at *g*. Take, therefore,

A+B+C : D::DK : *g*K,

or, A+B+C+D : D : DK+*g*K(D*g*) : *g*K, then K will be the center of gravity of A+B+C and D, and consequently the com-

* Since A : B :: BG : AG, by multiplying extremes and means A × AG=B× BG; i. e. when two bodies are in equilibrio, *the product of one of the bodies into its distance from the center of gravity, is equal to the product of the other body into its distance from the same center.* These quantities, A×AG, and B×BG, therefore express the respective forces by which A and B counteract each other's effects in their tendency to motion round G.

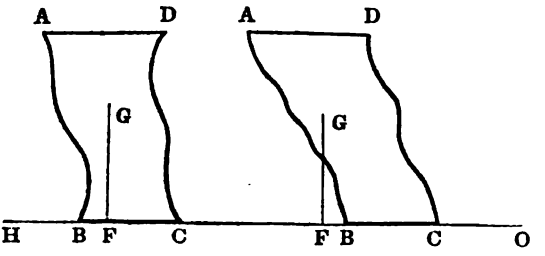
† Euc. v. 18.

mon center of gravity of the four bodies A, B, C, D; moreover, the pressure upon K will be the same as if A+B+C+D were placed at K; and thus we might proceed for any number of bodies.

It is evident that the foregoing demonstration does not at all depend upon the number or weight of the bodies, or their distance from each other; it rests merely on the supposition that their centers of gravity are connected together by inflexible rods void of gravity. It may therefore be applied to any number of particles of matter situated either in the same or in different planes, and placed at all possible distances from each other. Increase the number of these particles till they become contiguous to each other, and for the imaginary line void of gravity substitute the power of cohesion, then the system of bodies, A, B, C, D, &c. may represent an irregular mass of compact matter, not unlike such as are to be met with in the works of nature or of art; and although it may be difficult to find the actual center of gravity of such a mass, yet the latter part of our proposition still remains true; viz. that if this mass be supported, its pressure downwards will be the same as if the whole quantity of matter contained in it were concentrated in its center of gravity.

69. *Whatever be the form or dimensions of a body upon a plane parallel to the horizon, it will remain at rest, if the line drawn from its center of gravity perpendicular to the horizon* (*called* THE LINE OF DIRECTION,) *falls within its base.*

Fig. 42.

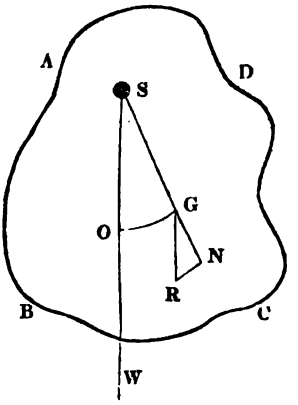


For let ABCD (Fig. 42.) represent the section of a body passing through its center of gravity G, and draw GF perpendicular to HO the plane upon which it stands; then, since the tendency of the body to descend is the same as if its whole weight were concentrated in G, it will rest or fall according as G is supported or not;

or, according as F falls within or without the base BC; moreover, the stability of the body will depend upon the distance at which the point F falls within the base.

70. *If a body be suspended freely from any point, it will not rest till the line which joins the center of gravity and the point of suspension, is perpendicular to the horizon.*

Fig. 43.



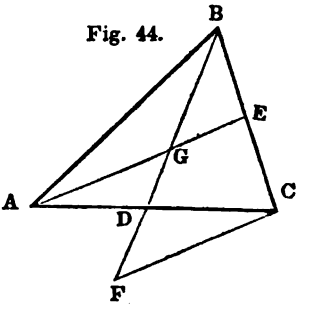
For let ABCD, represent the section of a body as before, G its center of gravity, S the point of suspension; join SG, and draw SOW perpendicular to the horizon; produce SG to N, and draw GR parallel to SW; then, since the weight of the body may be considered as collected in G, its tendency to motion will be along the line GR. Let GR therefore represent this tendency, which resolve into GN in the direction SG, and RN perpendicular to it; the part GN is counteracted by the reaction from the point of suspension S, and NR is employed in producing motion in the direction of the circular arc GO; G therefore (and consequently the body) will not remain at rest till NR vanishes, i. e. till the angle NGR (=OSG) vanishes, or SG coincides with SO.

Hence it follows, that if a body be suspended successively by different points, and perpendiculars to the horizon be drawn through the points of suspension, the center of gravity will lie in each of these perpendiculars, and consequently in the point of their intersection.

We proceed to apply the principles just now investigated, to the solution of a few practical examples.

71. *In a* TRIANGLE, *if a line be drawn from one of the angles bisecting the opposite side, the center of gravity of the triangle is in that line at the distance of* $\frac{1}{3}$ *of its length from the base.**

Fig. 44.



Bisect the side AC in D, and join BD, which will bisect all lines drawn parallel to AC; consequently, if BD be supported, the parts ABD, DBC of the triangle ABC will balance themselves on each side of it; hence the center of gravity is in the line BD. Bisect the side BC in E, and join AE; then, for the same reason as before, the center of gravity will be somewhere in the line AE; it must therefore be at their common intersection G. Produce now BD to F, and draw CF parallel to EA; then since BE=EC, BG will be equal to GF; but the two triangles AGD, DFC, have one side and two angles equal; $\therefore$GD=DF, and consequently GF (or BG) =2GD; hence BG=$\frac{2}{3}$BD, and GD=$\frac{1}{3}$BD.

72. *In a* TRAPEZOID, *the center of gravity is in the line that bisects the two opposite sides.*

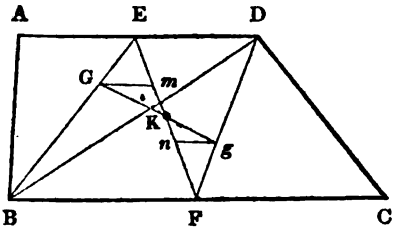
Let ABCD be a trapezoid and bisect AD, BC, in E, F, and join EF; then since EF bisects AD, BC, it will bisect all lines drawn parallel to BC,† and, consequently, the center of gravity of the tra-

* In finding the centers of gravity of plane figures, a lamina of matter of uniform density, in the shape of those figures is of course understood.

† For if BA, FE, CD be produced, they will meet in the same point, which will be the vertex of a triangle whose base is BC; and since EF bisects the base BC, it will bisect all lines drawn parallel to it.

pezoid is in the line EF. Join BE, BD, DF, and take $GE=\frac{1}{3}BE$, $gF=\frac{1}{3}DF$; then G is the center of gravity of the triangle ABD, (Art. 71.), and g the center of gravity of the triangle BDC. Join

Fig. 45.



Gg; then, conceiving the triangles ABD, BDC to be collected in G, g, their common center of gravity must be in the line Gg; i. e. the center of gravity of the trapezoid ABCD must be in the line Gg; it is also in the line EF; consequently it is in K, the intersection of EF and Gg. Draw Gm, gn, parallel to AD or BC; then since $EG=\frac{1}{3}BE$, Em must be equal to $\frac{1}{3}EF$; and for the same reason $Fn=\frac{1}{3}EF$; $\therefore Em=mn=nF$. Now K being the common center of gravity of the triangles ABD, BDC.

$$GK : Kg :: BDC : ABD :: BC : AD.$$
$$GK : Kg :: Km : Kn.\therefore$$
$$Km : Kn :: BC : AD.$$
$$Km : Km+Kn :: BC : BC+AD.$$

And, since $Km+Kn=mn=Em.\therefore$

$$Km+Em : mn :: 2BC+AD : BC+AD$$
$$EK : mn :: 2BC+AD : BC+AD \quad (1)$$
$$Km+Kn : Kn :: BC+AD : AD.$$

And since $mn=nF$, and $Kn+nF=FK.\therefore$

$$mn : Kn+nF :: BC+AD : BC+2AD$$
$$mn : FK :: BC+AD : BC+2AD \quad (2)$$

uniting (1) and (2).

$$EK : FK :: 2BC+AD : 2AD+BC.$$

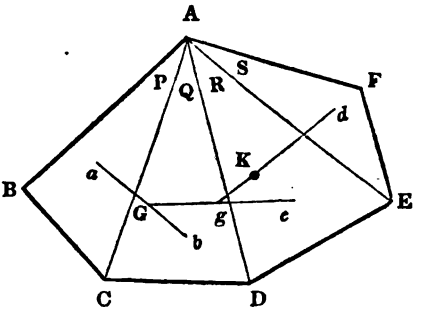
If, therefore, the line EF be divided in the ratio of the two last terms of this proportion, (formed of the known sides of the trapezoid,) it will give the center of gravity.

When $AD=0$, then the figure becomes a triangle, and $EK : FK :: 2BC : BC$; that is, $FK=\frac{1}{3}EF$, as was found by a different process in Article 71.

When AD=BC, the figure becomes a parallelogram, and EK : FK::3BC : 3BC; consequently, the center of gravity is in the center of the figure, as was shown in Article 65.

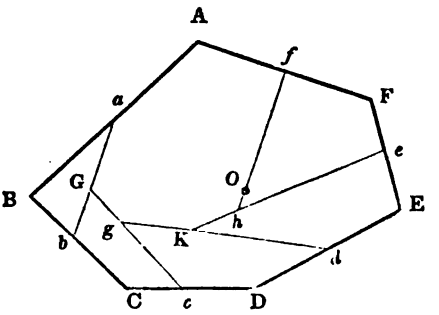
73. *The center of gravity of a* POLYGON *may be found by dividing the polygon into triangles, and finding the common center of gravity of these.*

Fig. 46.



Let ABCDEF be an irregular polygon, divided into triangles whose areas are represented by P, Q, R, &c., and whose centers of gravity are respectively *a*, *b*, *c*, &c. Conceive these triangles to be collected in the points *a*, *b*, *c*, &c.; join *ab*, and take *b*G : *a*G::P : Q, then G will be the center of gravity of the figure ABCD. Join G*c*, and take *cg* : G*g*::P+Q : R; then *g* will be the center of gravity of the figure ABCDE. Let *g* and *d* be joined, and make *d*K : *g*K :: P + Q + R : S, then K will be the center of gravity of the whole polygon; and so we might proceed, whatever be the number of sides.

Fig. 47.



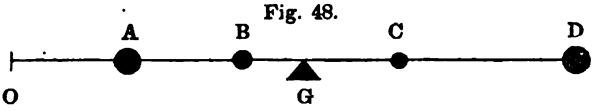
If it were required to find the center of gravity of the perimeter of the polygon; then bisect the sides in the points *a*, *b*, *c*, &c., (Fig.

47.) and (since the center of gravity of a right line is in its middle point) *a*, *b*, *c*, &c., will be the centers of gravity of the sides AB, BC, CD, &c., respectively. Join *ab*, and take *b*G : *a*G ::AB : BC; then G would be the center of gravity of that part of the perimeter represented by ABC. Again, join G*c*, and take *cg* : G*g*::AB+AC : CD, then *g* is the center of gravity of such part of the perimeter as is represented by ABCD; and so we might proceed till we had found the center of gravity of the whole perimeter.

74. *The distance from any assumed point of the common center of gravity of any number of bodies which have their centers of gravity in a right line passing through that point, is equal to the sum of the products arising from multiplying each body into its distance from the assumed point divided by the sum of the bodies.*

Let the bodies A, B, C, D, be so placed, that the line OD may pass through their respective centers of gravity; it is required to find their common center of gravity from any point O, in the line OD.

Fig. 48.



Suppose OD to be an inflexible line void of gravity, and let G be the common center of gravity of the bodies; then, if G be supported, the effort of each body to produce motion round G would be measured by the product of its weight into its distance from G; (Art. 67.) i. e., the effort of A=A×AG; of B=B×BG, &c.; and as the bodies are supposed to be in equilibrio about G, the sums of their efforts on each side G must be equal to each other, or

$$A\times AG+B\times BG=C\times CG+D\times DG, \text{ i. e., } A\times(OG-OA)+B\times(OG-OB)=C\times(OC-OG)+D\times(OD-OG).\therefore$$

$$A\times OG+B\times OG+C\times OG+D\times OG=A\times OA+B\times OB+C\times OC+D\times OD.$$

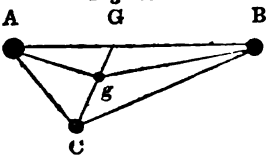
$$\text{Hence } OG=\frac{A\times OA+B\times OB+C\times OC+D\times OD}{A+B+C+D.}$$

75. Miscellaneous Examples.

1. *Three bodies*, A, B, C, *weighing respectively* 3, 2, *and* 1 *pounds, have their centers of gravity joined by the lines* AB, BC, CA; *of*

which AB=5 *feet*, BC=4, CA=2; *What is the distance of their common center of gravity from the body* C?

Fig. 49.



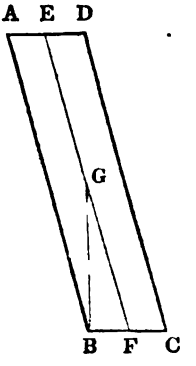
Since the three sides of the triangle ABC are 5, 4, 2, the three angles A, B, C, will be found, by the rules of Trigonometry, to be respectively 49° 27½′, 22° 20′, and 108° 12½′. Let G be the center of gravity of the bodies A and B, and g the common center of gravity of the three bodies found as in Art. 67; then, since AB=5, A=3, B=2, and A : B::BG : AG, AG will be equal to 2 feet, and BG to 3 feet; hence in the triangle GAC there is given AC=2, AG=2, and the angle CAG=49° 27½′, ∴ each of the angles AGC, ACG=65° 16¼′, from which CG is found to be equal to 1.673. But Cg : Gg::A+B : C, ∴ Cg : Cg+Gg(=CG=1.673) :: A +B (5) : A+B+C (6); hence $Cg=\frac{1.673\times 5}{6}=1.394$ feet.

2. *A cylindrical tower, consisting of uniform materials closely cemented together, is* 20 *feet high, and the diameter of its base is four feet; How far may it deviate from its perpendicular position, before it is in danger of falling?*

Let ABCD represent a section of the tower passing through its axis EF, and let G be its center of gravity. Suppose it to be so much inclined, that the perpendicular line GB, let fall from G, falls upon the edge of its base BC; then GF (10) : BF (2) : radius (1) : cos. GFB; ∴ cos. GFB $=\frac{2}{10}=.200=$cos. 78° 27′.

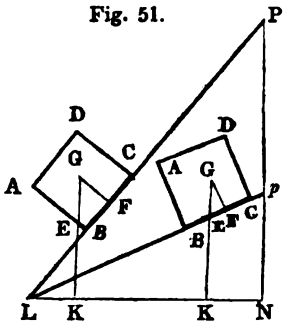
An angle of 78° 27′ is therefore the limit of its inclination, before it is in danger of falling. (Art. 69.) If the angle GFB is less than 78° 27′, then the perpendicular GB falls without the base, and the tower cannot sustain itself.

Fig. 50.



3. *A piece of timber of uniform density and prismatic form, a section of which (perpendicular to its sides, and passing through its center of gravity* G) *is represented by the square* ABCD, *is placed upon an inclined plane: It is required to show when it will have a tendency to roll, and when to slide down the plane.*

Fig. 51.



Draw GF perpendicular to the plane, and GK perpendicular to its base, and let PLN be greater and *p*LN less than half a right angle. In the former case, since ELK is greater than 45°, LEK or GEF will be less than 45° ; ∴ the angle EGF is greater than the angle GEF, and consequently EF is greater than GF or BF ; hence the body has always a tendency to fall over in the direction GE, and will therefore roll down the plane PL. In the latter case the angle EGF is less than GEF, ∴ EF is less than GF or BF ; the whole weight of the body, therefore, presses upon the plane *p*L. Let GE represent this weight, which resolve into two, GF, FE; GF will represent the reaction of the plane upon the body, and FE will represent a force which tends to make the body slide down the plane. Hence it appears, than the body will have a tendency either to roll or slide, according as the angle of the plane's inclination is greater or less than 45°.

In considering the circumstances under which a body would slide or roll down an inclined plane, it should be observed, that if the surfaces of the body and the plane be perfectly smooth, no rolling will take place, whatever be the angle or inclination of the plane. To give a body a tendency to rotary motion about its center of gravity (G), it is evident that there must be some mutual action between the surface of the body and the surface of the plane (such as that, for

instance, which arises from friction, the unrolling of rope, &c. &c.); if there be not some such action as this, all the parts of the body being equally accelerated, the body will, under all circumstances, slide down the plane.

76. Questions on the Center of Gravity.

1. *If three equal bodies be placed at the angles of any triangle; show that the common center of gravity of those bodies is in the same point with the center of gravity of the triangle.*

2. *Four bodies* A, B, C, D, *weighing respectively* 2, 3, 6, *and* 8 *pounds, are placed with their centers of gravity in a right line, at the distance of* 3, 5, 7, *and* 9 *feet from a given point: What is the distance of their common center of gravity from that given point; and between which two of the bodies does it lie?*

Ans. Between C and D; and its distance from the given point $7\frac{3}{11}$ feet.

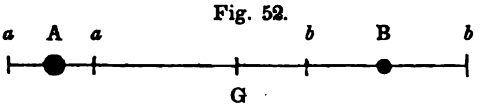
3. *The bodies* A, B, C, *weighing respectively* 5, 3, *and* 12 *pounds, are so placed, that* AB = 8 *feet,* AC = 4 *feet, and the angle* BAC *is a right angle: What is the distance of their common center of gravity from the body* C? Ans. 2 feet.

4. *Supposing the height of the cylinder in Exam.* 2. (Art. 75.) *to be only twice the diameter of its base: What is the limit of its angle of inclination before it is in danger of falling?* Ans. 60°.

Method of investigating the effect produced upon the common center of gravity of a system of bodies, when some or all of them are actually in motion.

77. *If two bodies approach to or recede from each other, with velocities inversely proportional to their weights, their common center of gravity will remain at rest.*

Fig. 52.



Let A and B be two unequal bodies; then if they approach to or recede from each other with velocities inversely proportional to

their weights, (in which case their momenta will be equal by Art. 14.) their common center of gravity G will remain at rest. For take $Aa : Bb :: B : A$, and suppose A to move through Aa whilst B moves through Bb, then (since $V \propto S$ when T is given) velocity of A : velocity of $B :: (Aa : Bb ::) B : A$ (Art. 67.); hence we have

$$AG : BG :: B : A,$$

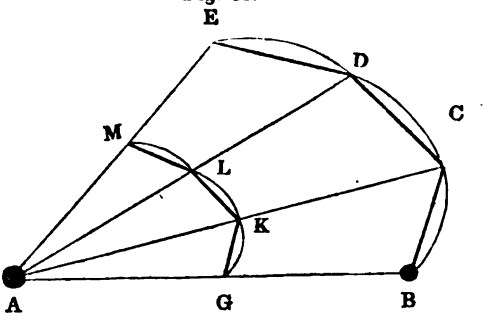
and $$Aa : Bb :: B : A;$$

$$\therefore^* AG \pm Aa : BG \pm Bb :: B : A, \text{ i. e. } aG : bG :: B : A;$$

from which it appears that G is their common center of gravity when the bodies are arrived at a and b, i. e. the center of gravity has remained at rest whilst the bodies have approached to or receded from each other through the spaces Aa, Bb.

78. *When one body moves uniformly, describing any figure around another at rest, the center of gravity of the two bodies describes a similar figure around the central body.*

Fig. 53.



Let A remain at rest, whilst B moves uniformly along the sides BC, CD, DE, of the polygon BCDE. When the body arrives at C, join AC, and take $AK : KC :: B : A$, or $AK : AC :: B : A+B$, then K will be the place of the center of gravity (Art. 68). When the body arrives at D, E, join also AD, AE, and divide them in the points L, M in the ratio of $B : A$, then will L, M be the position of the center of gravity at the end of those respective times. Let GK, KL, LM be now joined; and since $AG : AB :: B : A+B$, and $AK : AC :: B : A+B$, $\therefore AK : AC :: AG : AB$; hence GK is parallel to BC, and the triangle AGK similar to the triangle ABC.

* Algebra, 388.

In the same manner it may be proved that the triangles AKL, ALM are respectively similar to ACD, ADE, and the whole figure AGKLM to the polygon ABCDE. Whilst the body B therefore moves uniformly along the sides of the polygon BCDE, the common center of gravity G describes with a uniform motion a similar polygon GKLM; and since, from the nature of similar figures,* GK+KL +LM : BC+CD+DE:: (AG : AB, i. e.) B : A+B, the velocity of the center of gravity will be to the velocity of the body B::B : A+B. (Art. 12.) Suppose now the number of the sides of the polygon BCDE to be increased without limit, so that it may be considered as assuming the form of a Curve, then shall we come to this general conclusion, that, whilst the body B proceeds uniformly along the perimeter of the figure BCDE (whether rectilinear or curvilinear,) the center of gravity G will describe with a uniform motion a similar figure GKLM, with a velocity which is to that of B, as B : A+B.

79. *When a system of bodies are in motion, their common center of gravity will move in the same manner as if a body equal to the sum of the bodies were placed in that point, and the same motions were communicated to it as are communicated to the bodies separately.*

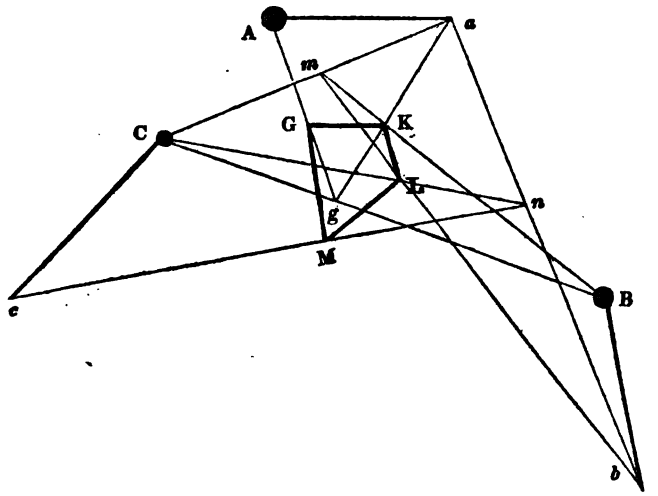
Let us take the case of three bodies, A, B, C, moving with uniform velocities, in equal successive parts of time, through the spaces A*a*, B*b*, C*c*. Let G be the position of the common center of gravity of the three bodies, and *g* that of B and C, before they begin to move; then (Art. 68.) G*g* : A*g*::A : A+B+C. Whilst A moves from A to *a*, B+C may be considered as at rest in *g*, ∴ (Art. 78.) the common center of gravity (G) will in the same time describe GK parallel to A*a*, and GK will be to A*a*::(G*g* : A*g*::)A : A+B +C. When A is arrived at *a*, join *a*C and BK; produce BK till it meets *a*C in *m*, then *m* will be the center of gravity of A and C;† join *mb*, then whilst B moves from B to *b*, the common cen-

* Algebra, 388.

† For, when A moves to *a*, the center of gravity of A and C is somewhere in the line C*a*. But when A moves to *a*, the center of gravity of the three bodies moves to K; therefore the center of gravity of A and C must also be in the line BK produced, since it must be such a point that A and C when placed there shall balance B. It must therefore be in the intersection of C*a* and BK, or at *m*.

ter of gravity will describe KL parallel to B*b*, and KL will be to B*b*::(*m*K : *m*B::)B : A+B+C. When B is arrived at *b*, join *ab*, CL; produce CL till it meets *ab* in *n*, then *n* will be the center of

Fig. 54.



gravity of A and B; join *nc*, then whilst C moves from C to *c*, the common center of gravity will describe LM parallel to C*c*, and LM will be to C*c*, ::(*n*L : *n*C::)C : A+B+C. Whilst the bodies A, B, C, therefore, in equal successive parts of time, move uniformly through the spaces A*a*, B*b*, C*c*, their common center of gravity will in the same time describe the polygon GKLM, whose sides GK, KL, LM, are respectively parallel to A*a*, B*b*, C*c*, and bear to them the ratio of A, B, and C : A+B+C.

80. If, instead of moving in successive intervals of time, the three bodies A, B, C, were all to begin to move at the same instant, and describe the lines A*a*, B*b*, C*c*, cotemporaneously; let us then consider what effect would be produced upon their common center of gravity. Now since GK, A*a*, are described in the same time, the velocity of the common center of gravity (*v*) : velocity of the body A (V)::GK : A*a*::A : A+B+C; hence A×V=(A+B+C)×*v*; i. e. the momentum of A (Art. 14.) is equal to the momentum of a body=A+B+C moving with the velocity (*v*); the

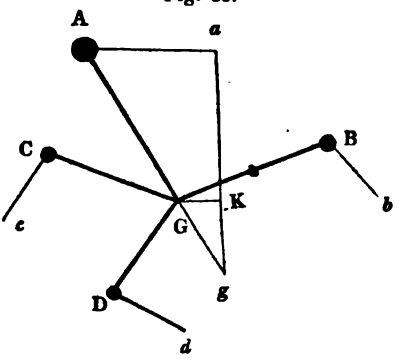
same force which causes the body A to move over A*a*, would in the same time cause a body equal to A+B+C to move over GK. For the same reason, the forces which impel B and C over B*b*, C*c*, are such as would in the same time cause a body =A+B+C to move over KL, LM. Hence it appears that the motion of the common center of gravity along the sides of the polygon GKLM is analogous to the motion of a body equal to A+B+C acted upon by three forces which would carry it over GK, KL, LM in the same time that they would carry the bodies A, B, C over the spaces A*a*, B*b*, C*c*, respectively. But a body acted upon at once by these forces would (Art. 45.) describe the other side GM of the polygon GKLM in the same time that it would describe either of the sides GK, KL, LM, when the forces act separately; if the bodies A, B, C, therefore, move cotemporaneously, their common center of gravity will describe the line GM, whilst the bodies themselves describe the three lines A*a*, B*b*, C*c**, and the same reasoning is applicable to any number of bodies.

81. Hence it appears, in the first place, that if the bodies which compose the system move uniformly in right lines, then their common center of gravity will either remain at rest, or will move uniformly in a right line; for if a body equal to the sum of the bodies were placed in that center, and then acted upon by the same forces which cause the bodies to move separately in right lines, it would either remain at rest, (viz. when the forces counteract each other,) or would describe uniformly the remaining side of a polygon whose other sides represent the quantity and direction of the several forces acting upon it. In the second place, it may be observed, that the common center of gravity of the system will not be affected by the mutual action of the bodies upon each other; for action and reaction being equal, the effect produced upon the common center of gravity by such mutual action, will only be that of two equal and opposite forces acting upon a body equal to the sum of the bodies placed in

* We have here supposed the bodies A, B, C, to have their centers of gravity in the same plane; in which case it is evident that the motion of their common center of gravity will be in the same plane. If the motion of the bodies be in different planes, then the value of each line GK, KL, LM might be found as before; but as they will then lie in different planes, the resulting quantity GM must be ascertained according to the principles laid down in Arts. 43 and 44.

that center; which would evidently not disturb its state, either of motion or quiescence. Lastly, it appears, that if the motion of the bodies in these right lines were to cease, and they were left to the mutual attraction of each other, then their common center of gravity would remain at rest, and the bodies would approach each other in lines drawn to it from their respective centers of gravity, and all collect together in that common center.

Fig. 55.

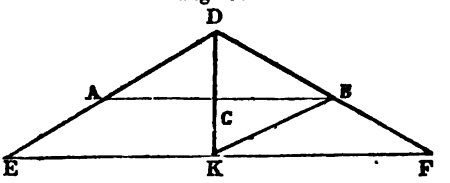


For if the motion of the bodies A, B, C, &c. in the direction A*a*, B*b*, C*c*, &c. were to cease, and they were left to the mutual attraction of each other, their common center of gravity, (G,) being under the same circumstances as a body (equal to the sum of the bodies) acted upon by equal and contrary forces, would remain at rest; but at the same time it is evident, that the bodies would move towards each other, and this motion we assert would be along the lines AG, BG, CG, &c. which meet in G. For suppose any one of them (A for instance) to move out of the line drawn from A to G, in the direction A*a*; then if *g* be the common center of gravity of the remaining bodies, (by Art. 78.) whilst A moves through A*a*, G will describe the line GK parallel to A*a*; and the same might be proved of the other bodies B, C, D, &c. In this case, therefore, the common center of gravity would not be at rest, which is contrary to the principle just now explained. Hence the bodies A, B, C, &c. in their approach towards each other, cannot possibly deviate from the lines AG, BG, CG, &c.; they will consequently all collect together at the point G.

82. Examples.

1. *Let two equal bodies* A *and* B, *move from the point* D, *with the same uniform velocity, along the sides* DE, DF *of the isosceles triangle* DEF, *whose angle* EDF = 120°; *It is required to compare the velocity of their common center of gravity with that of either of the bodies* A *or* B.

Fig. 56.

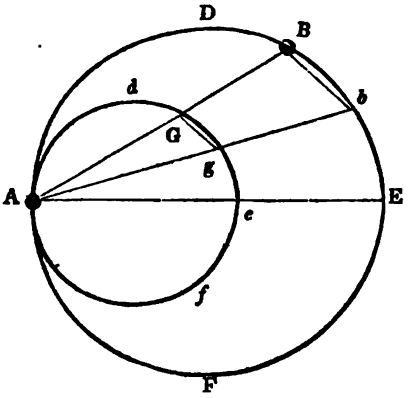


When the bodies are arrived at the points A, B, join AB; and since the bodies move with the same uniform velocity, DA=DB; ∴DA : DB::DE : DF, and AB is parallel to EF. Again, because A=B, the center of gravity G will bisect AB; hence, whilst the bodies move uniformly along DE, DF, the center of gravity will move through the line DK, which bisects AB, EF at right angles. Now EDF=120°, ∴EDK=60°, and DEK=30°; but since DE, DK are described in the same time, velocity of A : velocity of G::DE : DK::rad. : sin. 30°::2 : 1; the center of gravity, therefore, moves with half the velocity of either of the bodies A and B.

The same conclusion may be deduced from the principles laid down in Art. 79. of this section; for suppose the two bodies be placed at D, and take DB=BF; then if A remains at rest whilst B moves uniformly along DF, the center of gravity will in that time move from D to B; and if B remains at rest in F whilst A moves from D to E, the center of gravity will describe the line BK parallel to DE; and since the side DF is bisected in B, the side EF will be bisected in K. Suppose now the bodies A and B to move cotemporaneously, then their common center of gravity will describe DK, the third side of the triangle DBK, in the same time that it would have described either of the other two sides DB, BK when the bodies move successively.

2. *Let the two bodies* A *and* B *be placed at the extremity* A *of the diameter of the circle* ADF, *and then let* B *describe the circle* ADF *whilst* A *remains at rest in the point* A; *In what manner will their common center of gravity move?*

Fig. 57.

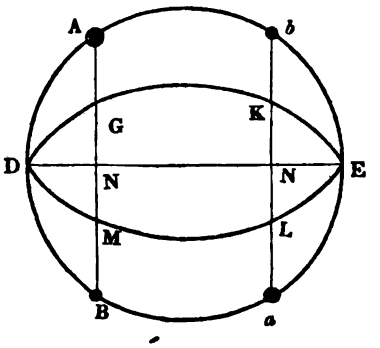


Let B, *b*, be any two positions of the body B, and G, *g*, the corresponding positions of the common center of gravity of A and B. Join, B*b*, G*g*; then, by Art. 78. G*g* is parallel to B*b*, $\therefore$AG*g*= AB*b*, hence AD*b*, A*dg*, are similar segments of circles; and when B has described the semicircle ADE, the center of gravity will have described the semicircle A*de*; and so for the semicircles on the other side of AE. Now A*e* : *e*E : : B : A, $\therefore$A*e* : AE : : B : A+B; hence while B describes the circle ADF, the common center of gravity of A and B will describe the circle A*df*, whose diameter : diameter of ADF : : B : A+B. This conclusion, indeed, follows immediately from the reasoning in Art. 78., for it was there shown that the whole figure described by the common center of gravity, is similar to that which the moving body (B) describes.

3. *Two bodies,* A *and* B, *begin to move in opposite directions at the same instant from the extremity* D *of the diameter* (DE) *of the circle* DAEB, *and continue to move on with the same uniform velocity till they meet in* E; *they pass each other at* E, *and then continue to move on till they arrive at the point* D, *whence they set off: What is the course of the common center of gravity during this revolution of the two bodies?*

Suppose the bodies arrived at the position AB, then since DA= DB, the line AB will be bisected by DE in N. Let G be the

Fig. 58.



common center of gravity of A and B, then A+B : B::AB : AG, ∴ (½A+B) : B::½AB (or AN) : AG; hence AN : AG:: A+B : 2B, and ∴ AN : AN−AG (GN) :: A+B : A+B−2B (A−B); i. e. AN : GN in the given ratio of A+B : A−B; consequently whilst the bodies A and B describe respectively the semicircles DAE, DBE, their common center of gravity describes the semi-ellipse DGKE; in the same manner it may be proved that whilst A and B describe the semicircles EBD, EAD, their common center of gravity would describe a semi-ellipse ELMD, equal and similar to DGKE. Whilst the bodies A and B therefore perform their respective revolutions, their common center of gravity will describe the ellipse DGEM, whose major axis : minor axis :: (AN : GN::) A+B : A−B.

Cor. If the bodies be equal, A − B=0, and the ellipse becomes a straight line. Indeed it is evident that in this case the common center of gravity would move in a line which always bisects AB, *ab*, i. e. in the diameter DE.

4. *Three bodies*, A, B, C, *at the same instant begin to move uniformly from the three angles of a given triangle, and in the same time change places in the direction* ABC: *How will their common center of gravity be affected by this motion of the bodies?*

Let G be their common center of gravity, and suppose the bodies first to move in succession. Whilst A moves from A to B, their

common center of gravity will (by Art. 78.) describe GK parallel to AB, and GK will be to AB::A : A+B+C, $\therefore$ GK$=\frac{A+AB}{A+B+C}$. Whilst B moves from B to C, the center of gravity will describe KL$=\frac{B+BC}{A+B+C}$; and if LM be drawn parallel to CA, and equal to $\frac{C\times AC}{A+B+C}$, whilst C moves from C to A, the center of gravity will describe LM. Suppose now the bodies to move cotemporaneously, then their common center of gravity will describe GM, (the remaining side of the polygon GKLM,) whilst the bodies change places in the direction ABC, (Art. 44.)

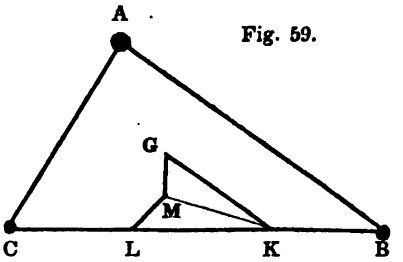


Fig. 59.

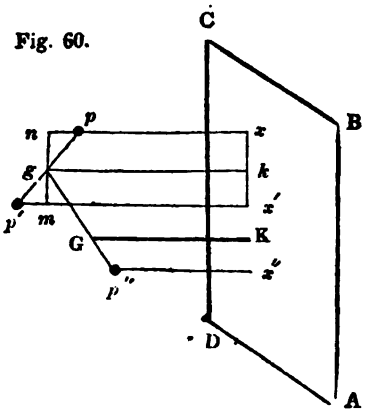
To find the actual value of GM, we have KL, LM, and the angle KLM (=ACB) given, from which MK and MKL may be found; but GKM=GKL, (or ABC)$-$MKL: in the triangle GKM there are therefore given GK, KM, and GKM, from which GM may be determined.

Cor. If A=B=C, then GK$=\frac{1}{3}$AB, KL$=\frac{1}{3}$BC, and LM$=\frac{1}{3}$AC, $\therefore$ GK, KL, LM, are to each other as AB, BC, AC; and since the angles GKL, KLM are respectively equal to ABC, BCA, the three lines GK, KL, LM will form a triangle similar to the triangle ABC. GM therefore in this case is equal to 0, and the body remains at rest. This also follows from the general theorem in Art. 79; for the common center of gravity being under the same circumstances as a body acted upon by three forces which are to each other as the three sides of a triangle taken in order, will, by Art. 55., remain at rest.

83. *The distance of the common center of gravity of any number of bodies or particles of matter from a plane given in position,*

is equal to the sum of the products arising from multiplying each body into its distance from the plane, divided by the sum of the bodies.

Fig. 60.



Let p, p', p'', &c. be any number of small bodies or particles of matter, and ABCD a plane placed in any position with respect to them. Join pp', and let g be the common center of gravity of p and p'; draw px, gk, $p'x'$ at right angles to the plane ABCD, and consequently parallel to each other; join xx', and since the points p, g, p' are in a straight line, the points x, k, x' will also be in a straight line, and therefore $x\,x'$ will pass through k. Join $g\,p''$, and let G be the common center of gravity of p, p', p''; draw GK, $p''\,x''$ perpendicular to the plane; and through g draw mn parallel to xx' meeting px produced in n.

Now $p : p' :: p'g : pg ::$ (by sim. triangles) $p'm : pn$;

$$\therefore p \times pn = p' \times p'm,$$

or $p \times (nx - px) = p' \times (p'x' - mx')$; But $nx = gk = mx'$,

$$\therefore p \times (gk - px) = p' \times (p'x' - gk,)$$

and $$(p+p') \times gk = p \times px + p' \times p'x',$$

$$\therefore gk = \frac{p \times px + p' \times p'x'}{p+p'}; \text{ for the same reason}$$

if $p+p'$ is placed at g, we have

$$GK = \frac{(p+p') \times gk + p'' \times p''x''}{(p+p') + p''}$$

$$= \frac{p \times px + p' \times p'x' + p'' \times p''x''}{p+p'+p''};$$

and thus we might proceed, whatever be the number of particles.

CHAPTER V.

OF THE COLLISION OF BODIES.

84. BODIES are divided into elastic and inelastic. Elastic bodies are such as, when compressed, restore themselves to their former state. Inelastic bodies are such as do not thus restore themselves. Thus, sponge, wool, cotton, and Indian rubber, are more or less elastic, and air, which restores itself with a force equal to that which compresses it, is perfectly elastic. But lead and clay are inelastic bodies, since, when they impinge upon one another, they do not rebound. Ivory, glass, and steel, are among the most elastic substances known. If we suspend two ivory balls by strings of the same length, and let them fall upon one another, (as in Fig. 4. page 16.) we may render the compression which they undergo on meeting apparent by dotting the points of contact with ink: after impact, these dots will be enlarged in a circular space around the original point.

Experiments on this subject are supposed to be made with two spheres or balls of the same density, moving uniformly in the line which joins their centers of gravity.

85. *When one* INELASTIC *body strikes upon another at rest, or moving with less velocity in the same direction, the two bodies move on together as one mass, with a velocity equal to the sum of the momenta divided by the sum of the bodies.*

Thus, let A, B represent the two bodies, and a, b, their respective velocities; then Aa will be the momentum of A, and Bb, that of B. (Art. 14.) The sum of their momenta is $Aa+Bb$. Let v be the common velocity after impact; then $(A+B)\times v=$ the momentum of the mass. Then $Aa+Bb=(A+B)\times v$. $\therefore v=\frac{Aa+Bb}{A+B}$.

If B is at rest, then the common velocity equals the momentum of A divided by the sum of the bodies: for then Bb becomes 0, and $v=\frac{Aa}{A+B}$.

The velocity lost by A equals the product of B into the DIFFERENCE of their velocities, divided by the sum of the bodies; and that gained by B, equals the product of A into the DIFFERENCE of the velocities divided by the sum of the bodies. For, the velocity lost by $A=a-v=a-\frac{Aa+Bb}{A+B}=\frac{B(a-b)}{A+B}$. The velocity gained by B $=v-b=\frac{Aa+Bb}{A+B}-b=\frac{A(a-b)}{A+B}$.

When B is at rest, these expressions become

$$\frac{Ba}{A+B} \text{ and } \frac{Aa}{A+B}.$$

86. *When the bodies move in* OPPOSITE *directions, the common velocity after impact equals the difference of their momenta, divided by the sum of the bodies.*

The momentum of the mass after impact is the difference of their momenta before impact. Hence $Aa-Bb=(A+B)\times v$ $\therefore v=\frac{Aa-Bb}{A+B}$.

The velocity lost by A equals the product of B into the SUM of the velocities divided by the sum of the bodies; and that gained by B equals the product of A into the SUM of the velocities, divided by the sum of the bodies.

For, the velocity lost by $A=a-v=a-\frac{Aa-Bb}{A+B}=\frac{B(a+b)}{A+B}$.

The velocity gained by B (in the direction of A) $=\frac{A(a+b)}{A+B}$.

When the two bodies are equal, and meet with equal velocities, the expression $v=\frac{Aa-Bb}{A+B}$ becomes $v=0$, and both bodies remain at rest. Since, in this case, $Aa=Bb$ $\therefore A:B::b:a$: therefore, conversely, *when bodies move, before impact, with velocities inversely proportional to their quantities of matter, they will be at rest after impact.* The same conclusion may be drawn from the consideration that, in this case, the bodies would meet with equal momenta. (See Art. 14.)

87. Examples for Inelastic Bodies.

1. A, *weighing* 3 *oz., and moving* 10 *feet per second, overtakes* B, *weighing* 2 *oz., and moving* 3 *feet per second: What is the common velocity after impact?* $7\frac{1}{5}$ feet per second.

2. *A weight of* 7 *oz., moving* 11 *feet per second, strikes upon another at rest weighing* 15 *oz.: Required the velocity after impact?* Ans. $3\frac{1}{2}$ feet per second.

3. A *weighs* 4 *and* B 2 *pounds; they meet in opposite directions,* A *with a velocity of* 9, *and* B *with one of* 5 *feet per second: What is the common velocity after impact?* Ans. $4\frac{1}{3}$ feet per second.

4. A=7 *pounds,* B=4 *pounds; they move in the same direction, with velocities of* 9 *and* 2 *feet per second: Required the velocity lost by* A *and gained by* B? Ans. A $2\frac{8}{11}$, B $4\frac{5}{11}$.

5. *A body moving* 7 *feet per second, meets another moving* 3 *feet per second, and thus loses half its momentum: What are the relative magnitudes of the two bodies?* Ans. A : B::13 : 7.

6. A *weighs* 6 *pounds and* B 5; B *is moving* 7 *feet per second, in the same direction as* A; *by collision* B's *velocity is doubled: What was* A's *velocity before impact?* Ans. $19\frac{1}{2}$ feet per second.

88. *In the collision of* ELASTIC *bodies, the velocity lost by the one and gained by the other, is twice that which it would have been, had the bodies been inelastic.*

According to the definition of elasticity, the body restores itself with a force equal to that which compresses it; consequently, as much momentum is exerted in the restitution as in the compression. In a given body, therefore, the velocity of restitution is equal to that of compression. Suppose, for example, that a ball of lead A, strikes upon another B; then what B gains A loses by reaction, and both bodies move on together; but when a ball of ivory (supposed perfectly elastic) impinges on another, it not only loses the momentum which it at first imparted to B, but the latter, in restoring itself after compression, exerts a force equal to that of reaction, and therefore destroys as much more of the motion of A. Again A, while receiving this second impulse from B, reacts with an equal force, and thus doubles the effect of its impulse upon B.

Distinguishing the corresponding elastic body by an accent, since, the direction being the same, the velocity lost by $A=\frac{B(a-b)}{A+B}$; $\therefore$ that lost by $A'=\frac{2B(a-b)}{A+B}$ $\therefore$ the velocity of A′ after impact $= a - \frac{2B(a-b)}{A+B}=\frac{(A-B)a+2Bb}{A+B}$.

So the velocity of B′ after impact $=\frac{(B-A)b+2Aa}{A+B}$.

When the directions are opposite,

The velocity lost by $A=\frac{B(a+b)}{A+B}$.

Do. do. by $A'=\frac{2B(a+b)}{A+B}$.

Hence, velocity of A′ after impact $=a-\frac{2B(a+b)}{A+B}=\frac{(A-B)a-2Bb}{A+B}$.

And velocity of B′ after impact $=\frac{(A-B)b+2Aa}{A+B}$.

89. *When equal elastic bodies impinge upon one another, each moves after impact with the previous velocity of the other body.*

For if B = A, then A − B or B − A are each equal to 0; $\therefore$ when the bodies move in the same direction before impact, the velocity of A after impact $=\left(\frac{2Bb}{A+B}=\frac{2Bb}{2B}=\right)b$; and the velocity of B after impact $=\left(\frac{2Aa}{A+B}=\frac{2Aa}{2A}=\right)a$. If they move before impact in opposite directions, then the velocity of A after impact $=\left(\frac{-2Bb}{A+B}=\frac{-2Bb}{2B}=\right)-b$, and the velocity of B after impact $=\left(\frac{2Aa}{A+B}=\frac{2Aa}{2A}=\right)a$. Hence, in all cases when the bodies are equal, they move after impact with interchanged velocities; that is, when the directions are the same, A moves on, after impact, with the velocity of B, and B moves on with the velocity of A; and when the directions are opposite, A returns with the velocity of B, and B returns with the original velocity of A.

90. *When equal elastic bodies, moving with equal velocities, in opposite directions, meet, each is reflected back with its original velocity.*

For, by Art. 89, the velocity of A after impact $=-b$, and that of $B=a$; and since $a=b$, each returns with its previous velocity.

If B rests before impact, then $b=0$; $\therefore$ the V_y of A after impact $=\frac{(A-B)a}{A+B}$, and velocity of B after impact$=\frac{2Aa}{A+B}$. If A be greater than B, then $\frac{(A-B)a}{A+B}$ is positive, $\therefore$ A moves after impact in the same direction as it did before with $V_y=\frac{(A-B)a}{A+B}$, and B precedes it with a velocity$=\frac{2Aa}{A+B}$(which is greater than a.) If A be less than B, then $\frac{(A-B)a}{A+B}$ is negative, $\therefore$ A is reflected back by its impact upon B with a velocity$=\frac{(A-B)a}{A+B}$; and B moves forward in A's original direction with a velocity$=\frac{2Aa}{A+B}$ (which is less than a.)

91. *If one elastic body strikes on another equal to it at rest, the first body will be brought to a state of rest, while the second will move on with the velocity of the first.*

If A be equal to B, then$\frac{(A-B)a}{A+B}$(which is the velocity of A after impact) $=0$, and $\frac{2Aa}{A+B}$ (which is the velocity of B after impact) $=\frac{2Aa}{2A}=a$; i. e. if B rests before impact, then A will rest after; and B will move forward in A's direction with A's velocity before impact.

Let there be a row of equal elastic balls A, B, C, &c. . . . X (Fig. 61.) placed contiguous to each other; then (by Art. 89.) if A be moved from its position and made to impinge upon B, it will rest after impact, and B will have a tendency to move on with A's

velocity; after the impact of B upon C, it will remain at rest, and C have a tendency to move on with A's velocity; after the impact of C upon D, it will remain at rest, and D will have a tendency to



Fig. 61.

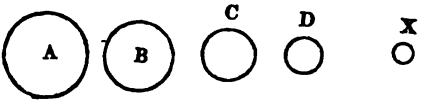


Fig. 62.

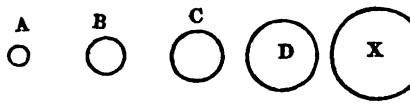


Fig. 63.

move on with the same velocity; and so the motion will be propagated through the whole row, and the last body X will move forward with the velocity of A, all the others remaining at rest.

If the bodies decrease in magnitude (Fig. 62.) then, since A is greater than B, (by Art. 90.) the velocity communicated to B, will be greater than that of A; and the velocity communicated from B to C greater than that of B, &c.; so that the last body will move forwards in the direction of A's motion with a velocity much greater than that of A, and the other bodies will follow it in such a manner, that the velocity of each succeeding body shall be greater than that of the preceding. On the contrary, if the bodies increase in magnitude, (Fig. 63.) since A is less than B, (by Art. 90.) the velocity communicated to B will be less than that of A, and A will be reflected back by B; for the same reason the velocity communicated from B to C will be less than that of B, and B will be reflected back by C; so that in this case all the bodies will move backwards except the last, and that will move forwards in the direction of A's original motion, but with a velocity much less than that of A.

92. *When a row of bodies are in geometrical progression, and the first impinges on the second, and motion is thus propagated through the series, the velocity of the first is to the velocity of the last,* as $1 : \left(\frac{2}{1+r}\right)^{n-1}$.

Let the series be $A, Ar, Ar^2 \ldots Ar^{n-1}$.

By Art. 90. when A impinges on B at rest the velocity communicated to B is $\frac{2Aa}{A+B}=\frac{2Aa}{A+Ar}=\frac{2a}{1+r}=b$.

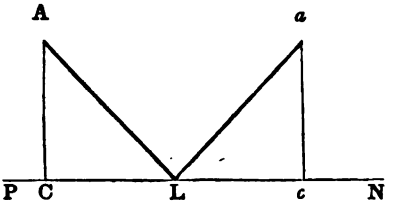
Again, the velocity imparted to C, is $\frac{2Bb}{B+C}=\frac{2Ar}{Ar+Ar^2}\times\frac{2a}{1+r}=\frac{2^2a}{(1+r)^2}$. Hence the successive velocities are $a, \frac{2a}{1+r}, \frac{2^2a}{(1+r)^2}$, &c. from which it appears that any term in the series is found by multiplying the original velocity by 2, raised to a power one less than the number of terms, and divided by $1+r$ raised to the same power. Consequently, the last term is $\frac{2^{n-1}a}{(1+r)^{n-1}}$. Hence,

V^y of the first : V_y of the last :: $a : \frac{2^{n-1}a}{(1+r)^{n-1}} :: 1 : \left(\frac{2}{1+r}\right)^{n-1}$.

93. *When a perfectly elastic body impinges on a perfectly smooth plane, it makes the angle of reflection equal to the angle of incidence.*

If a perfectly elastic body impinges perpendicularly upon a perfectly smooth plane, then, since the force of restitution is equal to the force of compression, it will ascend from the plane with the same velocity as that with which it impinged upon it. But if moving uniformly, it impinges upon the plane PN in the oblique direction AL,

Fig. 64.



then resolve AL into two AC, CL, of which, as in the former instance, the perpendicular part AC will not be destroyed, but will represent the velocity with which the body ascends from the plane; and CL will represent the velocity it has in the direction of the plane, the same as before. Take therefore Lc=LC, and from *c* draw *ca* at right angles to Lc and equal to CA, and join L*a*; then L*a* will represent the direction and velocity of the body after impact. But

since L*c*, *ca* are equal to LC, CA, and the angles L*ca*, LCA are right angles, L*a* will be equal to LA, and the angle *a*L*c*, to the angle ALC; hence the body will move after impact with the same velocity which it had before impact, and in a direction making the angle of reflection, equal to the angle of incidence.

94. Examples for Elastic Bodies.

1. A *weighing* 10 *lbs. and moving* 8 *feet per second, impinges on* B *weighing* 6 *lbs. and moving in the same direction,* 5 *feet per second: What are the velocities of* A *and* B *after impact?*

Ans. A's$=5\frac{3}{4}$. B's$=8\frac{3}{4}$.

2. A : B :: 4 : 3; *directions the same; velocities* 5 : 4. *What is the ratio of their velocities after impact?* Ans. 29 : 36.

3. A *weighing* 4 *lbs., velocity* 6, *meets* B *weighing* 8 *lbs., velocity* 4: *required their respective directions and velocities after collision?*

Ans. A is reflected back with a velocity of $7\frac{1}{3}$, and B, with a velocity of $2\frac{2}{3}$.

4. A *and* B *move in opposite directions;* A *equals* 4B, *and* $b=2a$: *How do the bodies move after collision?*

Ans. A returns with $\frac{1}{3}$, B with $1\frac{2}{3}$ its original velocity.

5. *There are ten bodies whose magnitudes increase geometrically by the constant ratio* 3, *and the first impinges on the second with the velocity of* 5 *feet per second: Required the motion of the last body?*

Ans. The last body would move with the velocity of $\frac{5}{3^{\frac{9}{2}}}$ feet per second.

CHAPTER VI.

OF THE LEVER.

95. In the preceding chapters, the motion of bodies has been supposed to arise either from collision, or from the immediate action of one or more forces. We now proceed to consider the effects produced, when these forces are made to act by the intervention of other

bodies. These intermediate bodies are called *Machines;* and by means of them the effect of a given force may be increased or diminished in any given ratio. Machines are divided into *simple* and *compound.*

96. The *simple* machines, or what are commonly called the MECHANICAL POWERS, are six in number; viz. 1. The *Lever;* 2. The *Wheel and Axle;* 3. The *Pulley;* 4. The *Inclined Plane;* 5. The *Screw;* 6. The *Wedge.* In philosophical strictness, the number of simple machines may be reduced to three; viz. the *lever,* the *inclined plane,* and the *cords* or *ropes* which connect the power and weight with the different parts of the machine; for the mechanism of the wheel and axle, and of the pulley, merely combines the principle of the lever with the tension of the cords; the properties of the screw depend entirely on those of the lever and the inclined plane; and the case of the wedge, so far as it is capable of mathematical demonstration, is very analogous to that of a body sustained between two inclined planes. *Compound* machines are formed from the combination of two or more simple ones. But it is not the object of this treatise to enter upon a full description of the nature and use of compound machinery; our intention is rather to explain upon mathematical principles, the general theory of mechanical action.

97. *The* LEVER *is an inflexible bar or rod, some point of which being supported, the rod itself is movable freely about that point as a center of motion.*

This center of motion is called the FULCRUM or PROP. When two forces act on one another by means of any machine, that which gives motion is called the POWER, that which receives it, the WEIGHT.

A lever is *straight,* when its arms (or the parts on each side of the fulcrum) are in one continued straight line; *bent,* when the two arms are straight, but make any angle with each other at the center of motion; and *crooked,* when one or both arms deviate from a straight line.

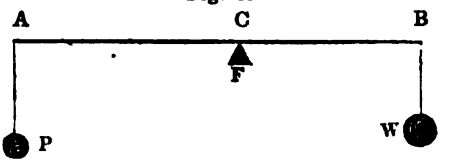
98. In treating of the Mechanical Powers, the first inquiry is, *What are the conditions of an equilibrium?* that is, When do the

power and weight exactly balance each other? This point being ascertained, any addition to the power, puts the weight in motion. The investigation first proceeds on the supposition that the action of the mechanical powers is not impeded by their own weight, or by friction and resistance, a suitable allowance being afterwards made for the various impediments.

We shall begin with estimating the relation between the forces acting upon the arms of a straight lever, which, of all the mechanical powers, is the most simple.

99. *If any two forces, acting in the same plane, and perpendicular to the extremities of a straight lever, be in equilibrio, they will be to each other inversely as the lengths of the arms upon which they respectively act.*

Fig. 65.



Let ACB be a straight lever, supported by a prop or fulcrum F, and movable about the point C as its fulcrum. From the extremities of its arms, CA, CB, let two weights, P, W, be suspended; and suppose them to be in equilibrio about C, the lever itself remaining in a horizontal position. In the present instance, let us also suppose that the lever AB, and the cords AP, BW, by which the weights are suspended, are entirely void of gravity; in which case it is evident that the equilibrium of the bodies does not at all depend upon the length of the cords AP, BW; and as (Art. 70.) the centers of gravity of the bodies P, W, are in the direction of the lines AP, BW, the effect will be the same whether the bodies be suspended by the the strings AP, BW, or whether they be placed with their centers of gravity in the points A, B, respectively.* In this latter case, the point C becomes the center of gravity of the weights P, W, and consequently P : W::BC : AC, (Art. 67.)

* For since the equilibrium does not at all depend upon the length of the lines AP, BW, we may suppose those lines to vanish; in which case the centers of gravity of P, W, may be considered as coinciding with the extremities A, B, of the lever.

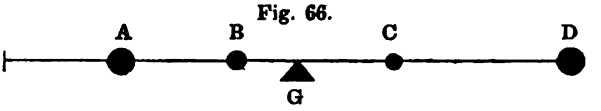
But it is evidently quite immaterial to the truth of the foregoing demonstration, whether the equilibrium of the lever be produced by the force of gravity of the two weights, P, W, or by the action of any other forces in the directions AP, BW.

100. *The effect of any forces to turn the lever about the center of motion is measured by the product arising from multiplying each force into the distance at which it acts from the fulcrum.*

For if the magnitude of the forces acting at A and B be represented by P and W respectively, then (since P : W::BC : AC) $P \times AC = W \times BC$; $\therefore P \times AC$ represents the effect of P, and $W \times BC$ represents the effect of W, to turn the lever round C.

101. *Any number of weights will keep each other in equilibrio upon the arms of a straight lever, when the sums of the products arising from multiplying each weight by its distance from the fulcrum, are equal on the two sides of that center.*

Fig. 66.

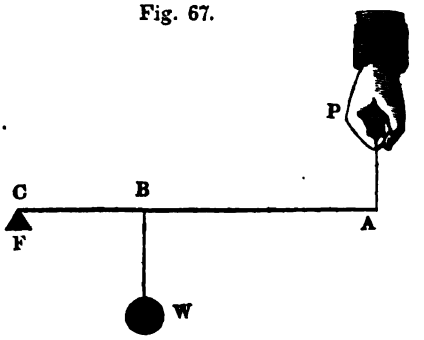


Let AD represent a straight lever whose fulcrum is G, and let the bodies or weights A, B, C, D, be placed upon its arms AG, DG, at different distances from G; then the effort of A to turn the lever about G being represented by $A \times AG$, of B by $B \times BG$, of C by $C \times CG$, &c., the whole effect upon the arm AG will be represented by $A \times AG + B \times BG$, and upon the arm DG by $C \times CG + D \times DG$; there will consequently be an equilibrium when $A \times AG + B \times BG = C \times CG + D \times DG$.

102. Levers are divided into three different orders, according to the position of the power and weight with respect to the fulcrum. I. In a lever of the first order, the fulcrum is between the power and weight, as in the preceding instance; and here the pressure on the fulcrum is equal to the sum of the weights. II. In a lever of the second order, the weight is placed between the power and the fulcrum, as in the annexed figure, where the weight W is supported by the power P acting upwards in the direction AP. In this case also

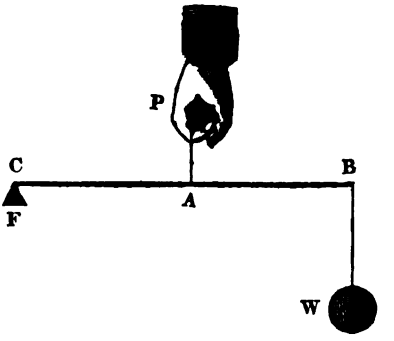
there is an equilibrium, when the power and weight are inversely as the arms on which they respectively act; for the effort of the weight W to turn the lever about C is measured by $W \times BC$ (Art. 68. Note); the effort of the power P (acting in the direction PA) to turn the lever about C, i. e., to sustain W, is measured by $P \times AC$; when there is an equilibrium, therefore, $P \times AC$ must be equal to $W \times BC$, or P : W::BC : AC, as before. Therefore, P is less than W; and the

Fig. 67.



pressure upon the fulcrum (P and W acting in opposite directions) is equal to W − P; for the pressure at A is the same as would be exerted on a fulcrum at that point, in which case the pressure on both points C and A would equal the whole weight W; therefore, the pressure on C equals W − P. III. In a lever of the third order, the

Fig. 68.



power acts between the weight and the fulcrum; but the equilibrium is produced on the same principle as before; for an equilibrium will take place when the opposite forces P and W are equal; which will

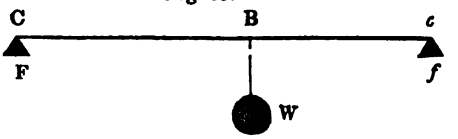
be when $P \times AC = W \times BC$; or when $P : W :: BC : AC$. Since BC is greater than AC, P is greater than W, and the pressure upwards from the fulcrum is represented by $P - W$.* Hence we have the following general principle, applicable to the three orders of levers,

When the forces act PERPENDICULARLY TO THE ARMS OF A STRAIGHT LEVER, *an equilibrium is produced, when the power is to the weight as the distance of the weight from the prop is to the distance of the power from the prop.*

In the first and second kinds of lever the weight is greater than the power; in the third kind less.

When a weight is sustained between two props, the part sustained by each prop is inversely as the distance of the weight from it.

Fig. 69.



For suppose C, *c* to be successively the centers of motion, then Press. on fulcrum *f*: weight W::BC : C*c*; for same reason, Weight W: Press. on fulcrum F:: C*c* : B*c*; ∴.

Press. on fulcrum *f*: Press. on fulcrum F::BC : B*c*; and as the whole weight is sustained by the two props, it is divided between them in the ratio of BC : B*c*.

103. Let us next estimate the relation of the forces which keep a lever in equilibrio, when its own weight is taken into consideration. Since this weight may be considered as collected in the center of gravity of the lever, (Art. 66.) its effective force is equal to the weight, multiplied into the distance of its center of gravity from the fulcrum. (Art. 100.) Suppose the lever to be of a cylindrical prismatic form, and that its weight $= w$, then,

In a lever of the first order, (Fig. 65.) since the center of gravity of the lever is in the middle point, and the distance of this point from the fulcrum $= \frac{1}{2}(AC - BC)$, the effect of the weight of the lever

* In this third order of levers, although the lever is supposed to move freely round the center of motion C, it is yet necessary to consider it as firmly connected with the prop at that point.

$=\frac{1}{2}w$ (AC$-$BC), and is exerted in the direction of the longer arm AC. In the case of an equilibrium, therefore, we have,

$$P\times AC+\tfrac{1}{2}w(AC-BC)=W\times BC,$$

$$\text{or } P=\frac{W\times BC-\frac{1}{2}w(AC-BC)}{AC}.$$

In a lever of the second order, (Fig. 67.) the whole weight of the lever operates in conjunction with W, and the distance of the center of gravity from the fulcrum in this case $=\frac{1}{2}$AC, $\therefore$

$$P\times AC=W\times BC+\tfrac{1}{2}w\times AC,$$

$$\text{or } P=\frac{W\times BC}{AC}+\tfrac{1}{2}w.$$

In a lever of the third order, (Fig. 68.) the whole weight of the lever operates in conjunction with W, and the distance of the center of gravity from the fulcrum $=\frac{1}{2}$BC, $\therefore$

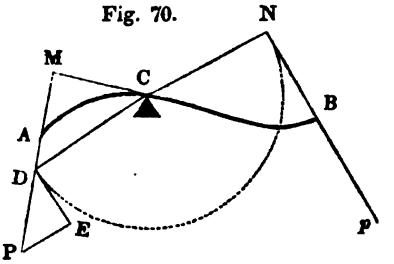
$$P\times AC=W\times BC+\tfrac{1}{2}w\times BC,$$

$$\text{or } P=\frac{(W+\frac{1}{2}w)\times BC}{AC}.$$

We have thus far confined our attention to the case in which the lever is supposed to be straight, and the forces to be applied at right angles to it; we now propose to take a more general view of the properties of the lever, whatever be its shape, or the directions of its forces.

104. *Two forces acting at the extremities of the arms of* ANY *lever will be in equilibrio, when they are to each other inversely as the perpendiculars let fall upon the lines of direction in which they respectively act.**

Fig. 70.



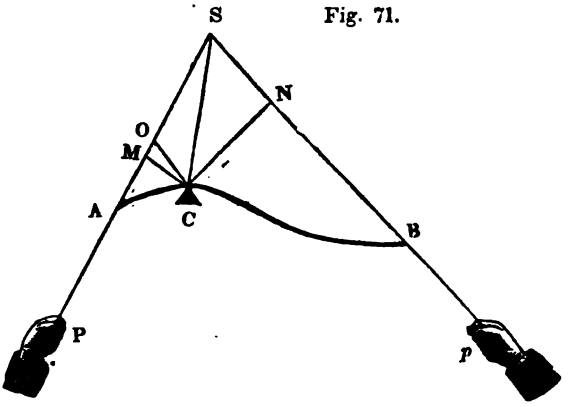
Let ACB (Fig. 70.) be any lever whose fulcrum is C; and let two forces P, p act in the directions AP, Bp, upon the extremities of

* This theorem evidently embraces the proposition in Art. 99.

its arms CA, CB. Produce PA, pB, to M, N, and let fall the perpendiculars CM, CN; with the longer perpendicular CN and center C describe the circular arc ND, and join CD. Let DP represent the magnitude of the force acting upon the lever at A in direction AP, and let it be resolved into two others, viz. DE perpendicular and EP parallel to the radius CD; then DE only is effectual to produce motion round the center C, the part EP being exerted merely to produce pressure upon the fulcrum in direction CD. Supposing, therefore, the lever to be perfectly inflexible, this effort of P to produce motion round C would be counteracted by a force equal to DE, applied perpendicularly at N, in direction NB. Let then the power p be equal to that part of P which is represented by DE, and (as it is indifferent in what point of the line of direction this power acts) conceive it to act at N. In this case the forces P, p will be in equilibrio; i. e. when P : p::PD : DE, the lever will be kept in equilibrio about the center of motion C; but by similar triangles PD : DE::CD(CN) : CM, ∴P : p::CN : CM.

105. Produce PA, pB till they meet in S; join CS, and draw CO parallel to pS, in which case the angle OCS=CSN. Now if CS be made radius, CM becomes the sine of CSO, and CN the

Fig. 71.

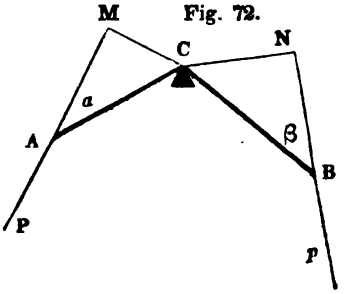


sine of CSN or OCS; but as SO : OC::sin. OCS(CN) : sin.CSO (CM); hence P : p::(CN : CM::)SO : OC. The two sides SO, OC of the triangle SOC represent, therefore, the relative magnitude and direction of the two forces, P, p; the third side SC will conse-

quently represent a force equivalent to them both (Art. 43.); and as this compound force acts directly towards C, the pressure upon the fulcrum will be represented, in quantity and direction, by the line SC.

106. If each arm of the lever be straight, but the two arms be inclined to each other in the given angle ACB, let $CAM=\alpha$, $CBN=\beta$, and rad.$=1$; then $AC : CM :: \text{rad.}(1) : \sin.\alpha$, $\therefore CM=AC\times\sin.\alpha$,

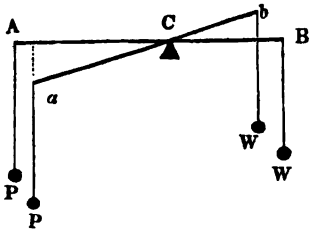
Fig. 72.



and $BC : CN :: \text{rad.}(1) : \sin.\beta$, $\therefore CN=BC\times\sin.\beta$; hence $P : p :: BC\times\sin.\beta : AC\times\sin.\alpha$. If $\sin.\alpha=\sin.\beta$, then $P : p :: BC : AC$, or P, p, are to each other inversely as the arms of the lever upon which they respectively act; which shows that the same law of equilibrium obtains in the bent as in the straight lever.

107. If the lever be straight, and the forces act parallel to each other, then $\sin.\alpha=\sin.\beta$, $\therefore P : W :: BC : AC$, as in Art. 106.;

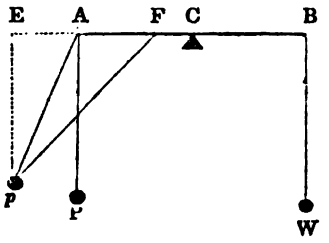
Fig 73.



and this will be the case whatever be the position of the lever; if therefore P and W be in equilibrio when the lever is in the horizontal position ACB, they will also be in equilibrio when it is in

any other position at CB; i. e. the lever thus acted upon will rest in any position.*

Fig. 74.



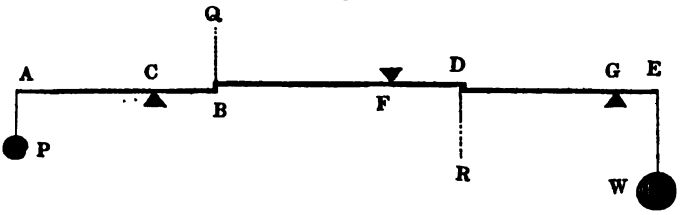
108. In the common balance or scales, the arms AC, CB, are equal to each other; ∴ when there is an equilibrium, P=W. But this equilibrium will be destroyed, if either P or W be removed from its perpendicular position. Suppose, for instance, a person placed in the scale P, was balanced by the weight W, but by pushing in the oblique direction *p*F against the arm CA, the scale was protruded into the position *p*A; then draw *p*E parallel to PA, and produce CA to meet it in E; and at the instant of the scales arriving at the position *p*A, the power will act at the perpendicular distance CE from the center of motion; its effect, therefore, (by Art. 100.) to turn the lever about C, will be measured by P×CE, i. e. its effect will exceed that of W in the proportion of CE : CA, and consequently the scale in which the person is, will preponderate, and the equilibrium be destroyed.

109. *In the* COMPOUND LEVER, *the opposite forces are in equilibrio, when the power is to the weight as the product of all the arms on the side of the weight is to the product of all the arms on the side of the power.*

* This also appears from Art. 104; for produce P*a* to M, then, when the lever is in position *a*C*b*, P : W : : CN : CM : : (by sim. triangles) C*b* : C*a* : : (since the arms of the lever are invariable) CB : CA. In thus asserting that the lever will rest in any position, it is of course taken for granted that the common center of gravity of P, W, and the lever coincides with the center of motion; for it is evident, from the principles laid down in Chap. IV., that the lever will only rest when that center is supported.

In a combination of levers connected with each other in the manner represented in the annexed figure, there will be an equilibrium

Fig. 75.



when $P : W :: BC \times DF \times EG : AC \times BF \times DG$. For suppose the equilibrium to exist, and that the forces which act at B, D, are represented by Q, R, respectively, then

$$P : Q :: BC : AC \text{ or } Q = \frac{P \times AC}{BC},$$

$$Q : R :: DF : BF$$

$$R : W :: EG : DG \text{ or } R = \frac{W \times EG}{DG};$$

$$\therefore P : W :: BC \times DF \times EG : AC \times BF \times DG. \qquad (A).$$

Also, the pressure on $C = P + Q = P + \frac{P \times AC}{BC} = \frac{P \times AB}{BC}$,

" " $F = Q + R = \frac{P \times AC}{BC} + \frac{W \times EG}{DG}$,

" " $G = R + W = \frac{W \times EG}{DG} + W = \frac{W \times DE}{DG}$.

We have here supposed the forces to act perpendicular to the extremities of the several levers; if they acted obliquely, or if the arms of the levers were inclined to each other, then for these arms must be substituted, in the proportion marked (A), the perpendiculars let fall from the centers of motion, C, F, G, upon the lines of direction in which the forces act. We now proceed to illustrate the foregoing theory by a few plain examples.

110. Examples.

1. *At the extremities of a straight lever, whose length is* 24 *inches, are placed two weights of* 5 *and* 7 *pounds: At what point*

must the fulcrum be placed so that these weights shall balance each other, the weight of the lever not being taken into the account?

This is the case of the lever of the first order, in which (Fig. 65.) P=5, W=7, and AB=24; and when there is an equilibrium, P (5) : W (7) :: BC : AC; $\therefore$ 12 : 7 :: BC+AC (AB=24) : AC= $\frac{168}{12}$=14; hence BC=24−14=10 inches.

2. *At the extremity of a lever of the second order, there acts a power which is of itself able to sustain only a weight of* 15 *pounds; but when acting under this mechanical advantage, it is able to sustain a weight of* 100 *pounds, placed* 5 *feet from it: What is the length of the lever?*

Referring to Fig. 67, 100 : 15 :: AC : AC − 5, $\therefore$ 100 : 85 :: AC : 5, $\therefore$ AC=$\frac{500}{85}$=5$\frac{15}{17}$ feet.

3. *A body suspended at the extremities of a balance whose arms are unequal, weighs* p *pounds at one end, and* q *pounds at the other: What is its real weight?*

A balance of this kind is called a false balance, because when the body is suspended at the extremity of the longer arm, the weight which balances it is above, and when suspended at the extremity of the shorter arm, is below, the true weight. But the true weight of the body is easily found by the following operation; viz. Let x=the true weight, and a the arm of the lever upon which p is suspended to balance it; and b the arm upon which q is suspended to balance it; then $x : p :: a : b$, or $x=\frac{pa}{b}$, and $x : q :: b : a$ or $x=\frac{qb}{a}$; multiply these two equations together, and we have $x^2=pq$ or $x=\sqrt{pq}$; i. e. the true weight of the body is a mean proportional between the apparent weights thus obtained. Hence to find the weight of a body by a false balance, we have this

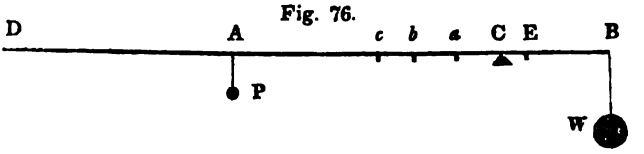
RULE.—*Take the weight of the body in each scale; multiply together the two weights thus found, and take the square root of the product.*

4. *To explain the construction of the steelyard.*

A steelyard is a lever of the first order, having two unequal arms BC, CD; a given weight P is movable along the longer arm CD,

so as to sustain a weight of a variable magnitude suspended from the extremity of a shorter arm CB.

Fig. 76.



1. Suppose the weight of the arms not to be taken into consideration, then $P : W :: BC : AC$ (Art. 99.) $\therefore W \times BC = P \times AC$, and as P and BC are given, $W \propto AC$. Hence, if Ca, ab, bc, &c. be taken equal to each other, (or $Cb = 2Ca$, $Cc = 3Ca$, &c.) then if P balances one pound when placed at a, it will balance two pounds at b, three pounds at c, &c.

2. Let us next suppose the steelyard to have weight, and that the excess of the weight of the longer arm CD above that of the shorter CB, is such that the movable weight P, when placed at E, would keep the arms in equilibrio; in which case this excess would be measured by $P \times CE$; if therefore a weight W, placed at B, be in equilibrio with the weight P placed at A, we should have $W \times BC = P \times AC + P \times CE = P(AC + CE) = P \times AE$; consequently, since P and BC are given, $W \propto AE$. The construction of the steelyard, therefore, would be the same as in the former instance, except that the graduation must begin from E, instead of from C.

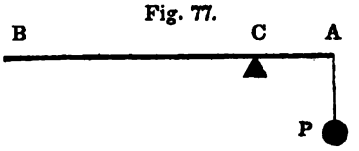
If the longer arm be divided into equal parts, to indicate the number of pounds, ounces, &c., which are contained in the variable weight W, the magnitude of the divisions may be found. For $W \times BC = P \times AE$; $\therefore \frac{AE}{W} = \frac{BC}{P}$, and as AE is the extent of the graduated arm corresponding to W, $\frac{AE}{W}$ or $\frac{BC}{P}$ will be the length of a division corresponding to 1 pound or 1 ounce, &c.

Hence, when only BC and P are given, the magnitude of each division will be equal to $\frac{BC}{P}$.

5. ACB *is a cylindrical straight lever whose weight is* (w,) *at the extremity* A, *a given weight* (P) *is suspended: It is required to de-*

termine the position of the fulcrum C, *so that* P *may be in equilibrio with the longer arm* BC.

Fig. 77.



Let AB$=a$,

AC$=x$,

Then the distance of the center of gravity of the lever from the fulcrum$=\frac{1}{2}a-x$.

Hence, from the principles laid down in Art. 103, when there is an equilibrium, we have

$$P\times x=w\,(\tfrac{1}{2}a-x),$$

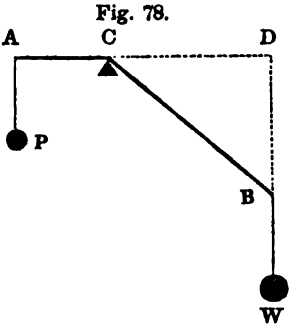
or $2Px+2wx=wa.$

$$x=\frac{wa}{2(P+w)}.$$

Cor. If $P=w$, then $x=\frac{1}{4}a$, or AC$=\frac{1}{4}$AB the length of the lever.

6. P *and* W *are suspended from the extremities of the arms of the bent lever* ABC, (*whose weight is not taken into consideration*): *It is required to find the angle of inclination* (ACB), *so that when there is an equilibrium,* AC *shall be parallel to the horizon.*

Fig. 78.



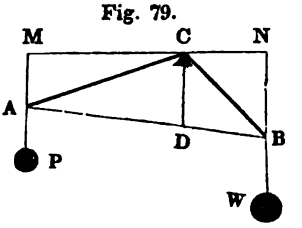
By Art. 106. P : W::BC$\times$sin.CBD : AC$\times$sin. PAC; but sin. PAC$=$rad.$=1$, and sin.CBD$=$cos.BCD; $\therefore$ P : W::BC$\times$cos. BCD : AC$\times 1$, hence cos.BCD$=\frac{P\times AC}{W\times BC}$, from which BCD and consequently ACB is known.

Cor. 1. If P=W, then cos. BCD=$\frac{AC}{BC}$; and if BC=2AC, then $\frac{AC}{BC}=\frac{1}{2}$; ∴ cos. BCD=$\frac{1}{2}$=cos. 60°; hence ACB=120°.

Cor. 2. If AC=BC, then cos. BCD=$\frac{P}{W}$; and if W=2P, then $\frac{P}{W}$ =$\frac{1}{2}$; ∴ cos. BCD=$\frac{1}{2}$=cos. 60°; hence, in this case also, ACB=120°.

7. *From the extremities of the arms* CA, CB, *of a bent lever, the weights* P, W, *are suspended: It is required to determine the position of the lever when these weights are in equilibrio.*

Fig. 79.



Through C draw MN parallel to the horizon, and produce PA, WB to meet it in M and N; by Art. 104. the lever will be in equilibrio, when P : W::CN : CM. Join AB, and draw CD parallel to MP or NW; CD will cut AB in the same ratio that it does MN*, i. e. DB : DA::CN : CM; hence DB : DA::P : W; from which it appears that if the line AB, which joins the extremities of the arms of the lever, be divided in the point D, in the ratio of P : W, and that point be brought immediately under the center of motion, it will give the position of the lever when P and W are in equilibrio.

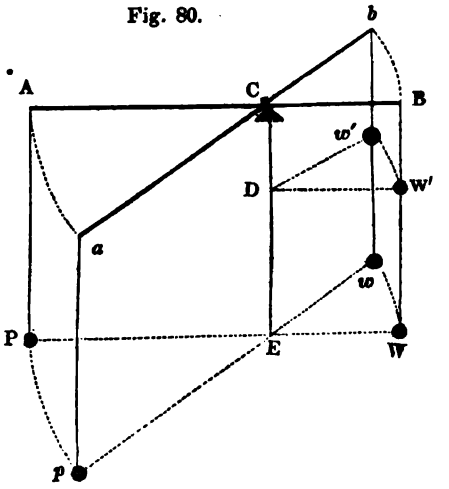
8. *If, from the arms of a straight lever* ACB, *two weights* P, W, *be suspended, and the lever be made to turn round its center of motion* C, *then will these two weights describe similar circular arcs, whether they be suspended by equal strings* AP, BW, *or by unequal strings* AP, BW′.

1. Let us first suppose them to be suspended by equal strings AP, BW, and let the lever be turned from its horizontal position ABC to any other position *aCb*; the weight P in the mean time describing

* Euc. vi. 2.

the line P*p*, and the weight W the line W*w*. Complete the parallelograms ACEP, CEWB, (which will evidently form one parallelogram APWB), and join E*p*, E*w*; then (since *ap*=AP=CE), *ap* and CE will be equal and parallel, ∴ C*ap*E* is a parallelogram.

Fig. 80.



In the same manner it may be proved that CE*wb* is a parallelogram, and that C*ap*E, CE*wb*, form one parallelogram *apwb*. Now E*p*=C*a*=CA=EP, ∴ P*p* is a circular arc, whose center is E, and radius EP or CA; for the same reason, W*w* is a circular arc, whose center is E, and radius EW or CB; and since the angle WE*w* is equal to the angle PE*p*, the arcs P*p*, W*w*, are concentric and similar.

2. Suppose now the string BW′ not to be equal to the string AP; then complete the parallelogram CBW′D, and (*w*′ being the position of W′ when the lever is in position *a*C*b*,) join D*w*′. In the same manner as before, it may be proved that W′ describes a circular arc W′*w*′ whose center is D, and radius DW′ or CB; but it is evident that DW′ is equal to EW, (being opposite sides of a parallelogram,) the arc W′*w*′ is therefore equal and similar to the arc W*w*; hence P and W describe similar circular arcs whilst the lever moves from the position ACB to the position *a*C*b*, whether the strings by which they are suspended be equal or unequal.

* Euc. i. 33.

111. Questions upon the Principles of the Lever.

1. *At one extremity of a straight lever whose length is* 7 *feet, a weight of* 10 *pounds is suspended; at the distance of* 5 *feet from the point of suspension a fulcrum is placed: What weight must be suspended from the other extremity of the lever, to keep it in equilibrio?* Ans. 25 pounds.

2. *A lever of the second order is* 25 *feet long: At what distance from the fulcrum must a weight of* 125 *pounds be placed, so that it may be supported by a power able to sustain* 60 *pounds, acting at the extremity of the lever?* Ans. 12 feet.

3. *A cylindrical straight lever is* 14 *feet long, and weighs* 6*lbs.* 5*oz.; its longer arm is* 9, *and its shorter* 5 *feet; at the extremity of its shorter arm a weight of* 15*lbs.* 2*oz. is suspended: What weight must be placed at the extremity of the longer arm to keep it in equilibrio?* (See Art. 103.) Ans. 7 pounds.

4. *A body weighs* 11 *pounds at one end of a false balance, and* 17*lbs.* 3*oz. at the other: What is its real weight?*

Ans. 13lbs. and 12oz.

5. A *and* B *are of the same height, and sustain upon their shoulders a weight of* 150 *pounds, placed on a pole* $9\frac{1}{3}$ *feet long; the weight is placed* $6\frac{2}{3}$ *feet from* A: *What is the weight sustained by each person?* (See Art. 102.)

Ans. A sustains $42\frac{6}{7}$ pounds, and B sustains $107\frac{1}{7}$ pounds.

6. *The longer arm of a steelyard is* 2 *feet* 2 *inches in length, and the shorter* $2\frac{3}{4}$ *inches; and its apparatus of hooks, &c. is so contrived, that a weight of* 2 *pounds placed upon the longer arm, at the distance of* 10 *inches from the center of motion, will balance* 8 *pounds placed at the extremity of the shorter arm; the movable weight (of* 2 *pounds) cannot conveniently be placed nearer to the fulcrum than* $\frac{3}{8}$ *of an inch: What must be the graduation of the steelyard that it may weigh ounces, and what will be the greatest and least weights that can be ascertained by it?*

Ans. The graduation is to 12ths of an inch; and it will weigh from 1 to 20 pounds.

7. *The arms of a straight lever are to each other as* 7 : 9, *and it is acted upon obliquely by two forces; the force* (P) *applied at*

the extremity of the longer arm, is inclined to it at an angle of 50°, *and* (*p*) *at the shorter at an angle of* 80°: *What is the proportion between the forces, when the lever is in equilibrio?* (See Art. 106.)

Ans. P=*p*.

8. *The arms of a bent lever are equal, and* P : W : : 1 : 2: *What must be the inclination of the arms to each other, that the arm from whose extremity* P *is suspended may be parallel to the horizon?* Ans. 120°.

9. *In a combination of levers connected together in the manner represented in Fig.* 75. *the three shorter arms* (BC, DF, EG,) *are respectively* 2, 5, *and* 3 *feet; the three longer arms* (AC, BF, DG,) *are* 13, 14, *and* 15 *feet; the weight* (P) *suspended from* A *is* 5 *pounds: What weight will it sustain at* E? Ans. 455 pounds.

CHAPTER VII.

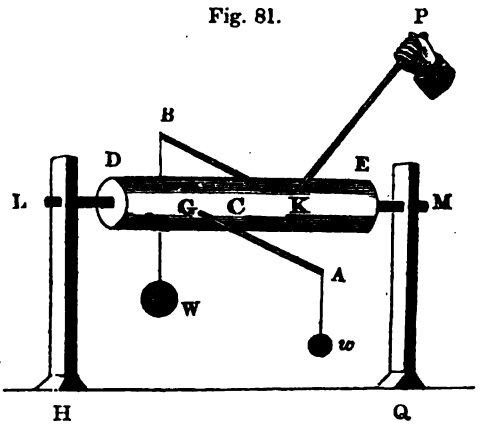
OF THE WHEEL AND AXLE; AND THE PULLEY.

WHEEL AND AXLE.

112. In order to explain the manner in which the wheel and axle operate upon each other, suppose DE to be a cylindric roller supported upon the props LH, MQ, and movable about the axis LM. Let two straight inflexible rods AG, BC be inserted into this cylinder, in a direction perpendicular to the axis, but parallel to each other and the horizon; let there be another rod PK perpendicular to the axis, but making any angle with the plane passing through BC or AG, and the axis. From the extremities of the rods BC, AG let the weights W, *w* be suspended; then (Art. 100,) W × BC will represent the effect of W, and *w* × AG the effect of *w*, to turn the roller about the axis LM; and supposing the rods AG, BC, and the roller DE, to be perfectly rigid and inflexible, it is evident that these two weights will counteract each other's effects, in the same manner as if they were acting at the extremities of the arms of a straight lever. When W and *w* therefore are in equilibrio, W × BC will be equal to *w* × AG, or *w* : W : : BC : AG; and this will be the case what-

ever be the length of the rod BC. Suppose that rod to be equal to the radius of the roller, then the string BW will become a tangent to the roller, and the foregoing proportion becomes w : W :: the radius of the roller : AG. Let us next suppose the weight W to be

Fig. 81.



kept in equilibrio by a power P acting at right angles to the extremity of the rod PK; then may P and W be considered as acting at the extremities of the arms PK, BC of the bent lever; and since they act at right angles to those arms, (Art. 106.) P : W :: BC : PK;* and (when BW acts as a tangent to the roller,) P : W :: radius of the roller : PK.

113. *In the wheel and axle an equilibrium is produced when the power acting at the circumference of the wheel : weight sustained upon the axle :: radius of the axle : the radius of the wheel.*†

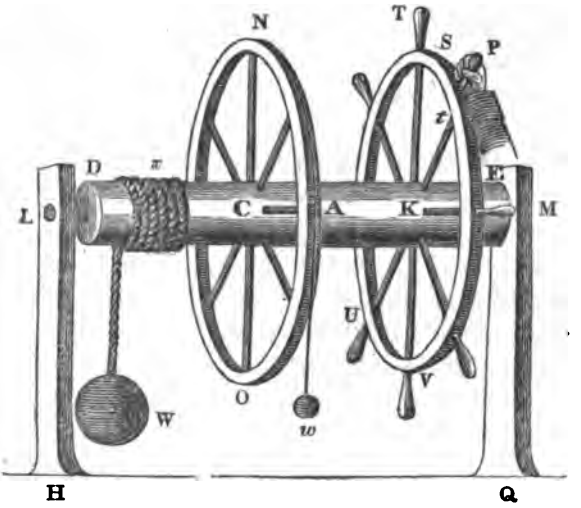
For, let the weight W (which is suspended by a rope going round the axle DE,) be kept in equilibrio, either by another weight (w) suspended from a rope going freely round the wheel NAO, or by a power P acting at right angles to the handles TS, Pt, &c. of the wheel StV, and let the planes of these wheels be at right angles to the axis LM of the machine; then, in the former case, the weight

* For in this case, sin. a= sin. β.

† Let R= radius of the wheel, r= radius of the axle, then P : W :: r : R, $\therefore$ P × R=W × r; if W and r be given, and P and R variable, then P $\propto \frac{1}{R}$; i. e. to sustain a given weight upon a given axle, the power must be increased as the radius of the wheel is diminished; and vice versa.

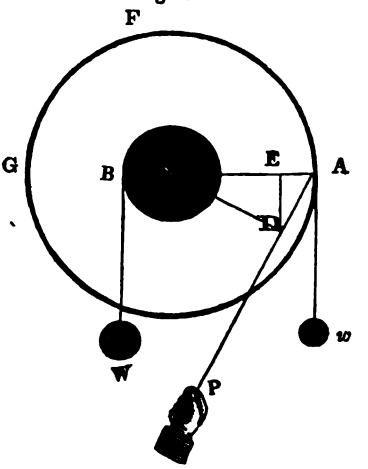
W may be considered as sustained by a weight (w) acting at right angles to the extremity of the arm AC of a straight lever, and in the latter, by a power (P) acting at right angles to the extremity of the

Fig. 82.



arm tK of a bent lever, the weight itself in each case acting at the distance of the radius of the axle from the center of motion; hence, by (Art. 112.) w or P : W : : radius of the axle : AC or tK.

Fig. 83.



114. If the power does not act at right angles to the radius of the wheel, but in some oblique direction, as AP (in the above figure,

which represents a section of the wheel and axle,) then let fall CD perpendicular to AP. By the property of the lever, P and W are to each other inversely as the perpendiculars let fall from the center of motion upon the lines of direction in which they respectively act, (Art. 104.); in this case, therefore, P : W::CB : CD:: radius of the axle : radius of the wheel × sine of the angle which P makes with the radius of the wheel.

115. Hitherto we have not considered the thickness of the rope; when that is taken into the account, we must add the half of it to the distance at which W and w respectively act.* Let therefore $2t$ = diameter of the rope, and let R = rad. of the wheel, r = rad. of the axle; then if the thickness of the rope be taken into consideration, we have w : W::$r+t$: R+t; and since in this case the same quantity (t) is added to each term of the ratio r : R, w must bear a greater ratio to W than that of r : R, or of the radius of the axle to the radius of the wheel.†

116. In a combination of wheels, such as is represented in the following figure, where a power (P) acts upon the winch or handle PQ,‡ which turns the wheel A, which acts upon the wheel B, from which the motion is propagated through the wheels C and D to the axle E, about which the rope that sustains the weight (W) is wound, let the force exerted by the wheel A upon the wheel B = p, by C upon D = q; then, supposing P and W to be in equilibrio, we have,

* For W, w act in the direction of the axis of the rope, and this axis is evidently removed from the circumference of the wheel or axle by half the thickness of the rope.

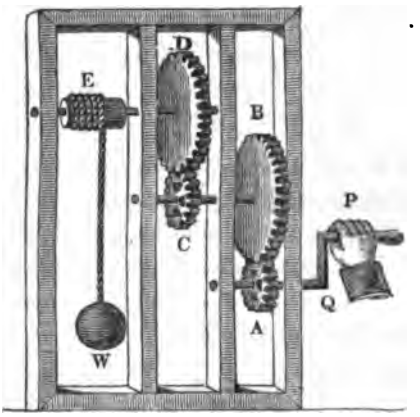
† In this article we have considered the ropes which go round the wheel and axle to be of the same thickness, and that it coils round the axle but once. But suppose the thickness of the rope to which W is appended to be 2T, that of w to be $2t$, and the rope to coil round the axle any number of times denoted by n; then it is evident, that for each coil of the rope after the first, W will be farther removed from the circumference of the axle by the whole thickness (2T) of the rope; the most general form therefore under which the relation of w: W can be exhibited, when the thickness of the rope is taken into consideration, is, w : W :: r+(2n—1)T : R+t.

‡ In this case the effect will evidently be the same as if the power acted at the circumference of a wheel whose radius is PQ.

$P : p :: \text{rad. of wheel A } (r) : PQ,$
$p : q :: \text{rad. of wheel C } (r') : \text{rad of wheel B } (R),$
$q : W :: \text{rad. of axle } (a) : \text{rad. of wheel D } (R');$
$\therefore P : W :: a \times r \times r' : PQ \times R \times R'$*;

where the demonstration goes upon the same principle as that of a combination of levers, in Art. 109.

Fig. 84.



117. Instead of the power being applied to the handle or winch PQ, suppose it to be applied at the circumference of a wheel whose radius is (R); let the radius of the axle $= r$, the radii of the small wheels $= r'$, r'', &c., and of the larger ones $= R'$, R'', &c.; then, whatever be the number of such wheels, the proportion expressing the relation between P and W, when there is an equilibrium, will be $P : W :: r \times r' \times r''$ &c. $: R \times R' \times R''$ &c. :: product of the radii of all the smaller wheels (or axles) : product of the radii of all the larger ones.

If the radii of the large wheels are equal to each other, and also those of the small wheels, then $P : W :: r^n : R^n$, where n equals the number of wheels or axles.

118. Examples.

1. *What must be the diameter of a wheel by which a weight of* 100 *pounds suspended by a rope going round an axle, whose radius*

* Algebra, Art. 392.

is 6 *inches, may be kept in equilibrio by a power acting upon it equivalent to* 12 *pounds?*

Let r = radius of the axle,
x = do. do. wheel,

then, by Art. 113,
$P(12) : W(100) :: r(6) : x$,
$\therefore x = \frac{600}{12} = 50$ inches $=$ *Ft.* 4 *In.* 2
and the diameter $= 8$ 4.

2. *A weight of* 500 *pounds is sustained by a rope of one inch diameter, going round an axle whose radius is* 8 *inches; and the power acts close to the circumference of a wheel whose radius is* 4 *feet: What is the ratio of* P : W?

This is a case of Art. 113, where the weight is not kept in equilibrio by another weight, but by some power acting upon a handle close to the circumference of the wheel, (as P acts upon the wheel S*t*V in Fig. 82.), and since t disappears in the 4th term of the proportion, w (or P) : W :: $r+t$: R$+t$, which becomes P : W :: $r+t$: R. In the present instance, W $= 500$, $r = 8$ inches, $t = \frac{1}{2}$ inch, R $= 48$ inches, $\therefore$ P : 500 :: $8\frac{1}{2}$: 48; or $P = \frac{4250}{48} = 88.54$ pounds.*

3. *In Fig.* 84, PQ $= 1$ *foot; the radii of the wheels* A, C *are each* 4 *inches; the radii of the wheels* B, D *are each* 15 *inches; and the radius of the axle* E *is* 3 *inches: What power must be applied to* P *to support a weight of* 600 *pounds?*

By Art. 185, P : W (600) $:: a \times r \times r' : PQ \times R \times R'$,
$:: 3 \times 4 \times 4 : 12 \times 15 \times 15$,
$:: 4 : 225$, $\therefore P = \frac{2400}{225} = 10\frac{2}{3}$ lbs.

119. Questions upon the Principles of the Wheel and Axle.

1. *A power of* 14 *pounds acts upon a wheel whose diameter is* 9 *feet: What weight will keep it in equilibrio, supposing the rope which*

* If the thickness of the rope be not considered, then P : W (500) : : r (8) : R (48), $\therefore P = \frac{4000}{48} = 83.33$; it makes a difference, therefore, of 5.21 lbs., whether this thickness be or be not taken into the account.

supports that weight to be wound round an axle whose diameter is 7 *inches?* Ans. 216 pounds.

2. *A power of* 4 *pounds keeps in equilibrio a weight of* 176 *pounds, by means of a wheel whose diameter is* 11 *feet: What is the diameter of the axle?* Ans. 3 inches.

3. *A power* (P) *acting by means of a rope going over a wheel whose diameter is* 7 *feet* 11 *inches, supports a weight of* 528 *pounds; the diameter of the axle is* 9 *inches, and the rope by which* P *and* W *are suspended is* 2 *inches thick: What must be the magnitude of* P, *supposing the thickness of the rope to be taken into consideration?* Ans. P=55 pounds.

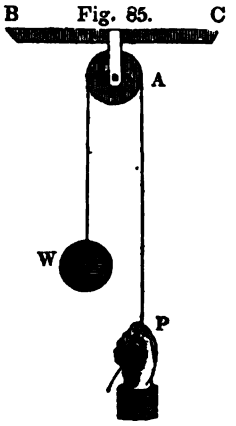
4. *Four wheels,* A, B, C, D, *whose diameters are* 5, 6, 10, *and* 2 *feet respectively, are put in motion by a power of* 15 *pounds applied at the circumference of the wheel* A; *these wheels act upon each other by means of three smaller wheels, the diameter of each of which is* 10 *inches; the last wheel* D *turns an axle whose diameter is* 4 *inches: What weight may be sustained by a rope going over this axle?* Ans. 46,656 pounds.

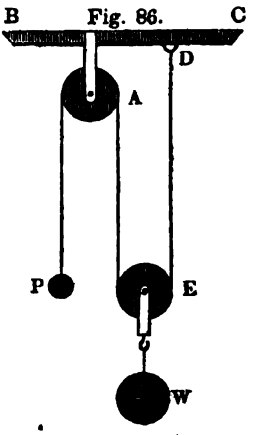
THE PULLEY.

120. A pulley is a small grooved wheel movable about a pivot, the pivot itself being at the same time either fixed or movable. The principle upon which a weight is sustained by means of a pulley or system of pulleys, is very simple, and will be readily understood from the following investigation.

In the single fixed pulley A, (Fig. 85.) about which the weight W is sustained by the power P acting on a string WAP passing along the groove in the circumference, no mechanical advantage is gained; for since the rope passes freely round the pulley, it is evident that the tension on each side of it must be the same, and consequently that the power must be equal to the weight which it sustains. The only advantage attending a pulley of this kind is, that a given power may be made to sustain or put in motion a given weight in a more convenient manner, by altering at pleasure the direction in which the power acts. The pressure upon the pivot or axis of the pulley A is evidently equal to P+W.

121. But if a weight W (Fig. 86.) be sustained by a power P acting on a string going over a movable pulley E as well as the fixed one A, then it is evident that this weight is sustained by two strings AE, DE; and as it is suspended from the center of the pulley E, these





ropes must act at equal distances from that center; consequently, each string must sustain half the weight. But it is evident, that whatever is the weight sustained by the string AE, the same must be sustained by the power P, which acts upon a string going freely over the fixed pulley A; hence, when there is an equilibrium, $P=\frac{1}{2}W$, or $W=2P$, $\therefore P : W :: 1 : 2$. With respect to the pressure upon the hook D, it is $\frac{1}{2}W$ or P, and upon the axis of the pulley A it is equal to $P+\frac{1}{2}W=2P$.

122. The same principle applies to the system of pulleys, in which the same string goes round all the pulleys, as in Fig. 87. For it is evident that the weight W is supported by all the strings at the lower block; if therefore the number of these strings be (n), each string must support $\frac{1}{n}$th part of the weight. But when there is an equilibrium, whatever be the tension upon each of these strings which support the weight, the same will be the tension on the string upon which the power P acts; hence $P=\frac{1}{n}W$, or $W=nP$, $\therefore P : W :: 1 : n$* (where n=number of strings at the lower block, or twice the

* If two blocks of pulleys of this kind (in which m and n are respectively the number of strings,) were combined together, so that the effect (E) produced by the first block should act as power upon the second, then $P : E :: 1 : m$, and $E : W :: 1 : n$; $\therefore P : W :: 1 : mn$.

number of movable pulleys.) The pressure upon the hook B or C is evidently equal to $P+W=P+nP=(n+1)P$.

Fig. 87.

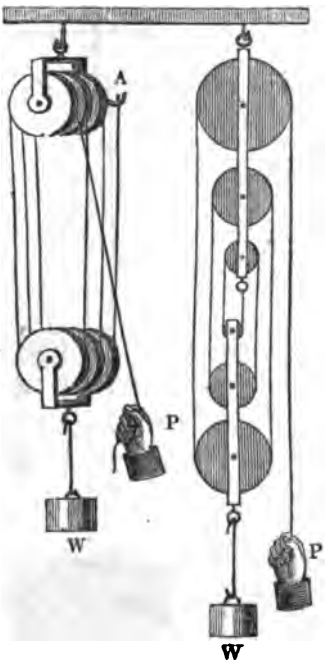
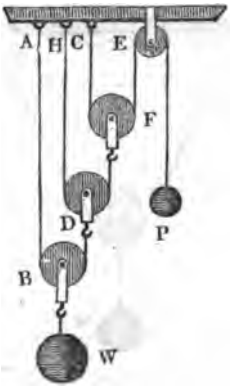


Fig. 88.



123. When the same string does not go round all the pulleys, but each pulley (Fig. 88.) has a separate string CFE, HDF, ABD, &c., going round it, and fastened to the hooks A, H, C, &c., then the relation between P and W must be estimated by a different method. Thus (since the string CFE goes over a single movable pulley) by Art. 121, P : weight sustained by pulley F ::1 : 2; and weight sustained by F : weight sustained by D ::1 : 2,

weight sustained by D : weight sustained by B i.e. W ::1 : 2,

$\therefore P : W :: 1 : 2\times2\times2$, &c., $:: 1 : 2^n$, or $W=2^nP$ (if n be the number of movable pulleys). In this system of pulleys, the pressure upon the hook $A=\frac{1}{2}W=\frac{1}{2}\times2^nP=2^{n-1}P$; upon the hook H $(=\frac{1}{2}$pressure upon A$)=\frac{1}{2}\times2^{n-1}P=2^{n-2}P$, &c.; and the pressure upon pulley $E=2P$.

124. Hitherto we have considered the strings as acting parallel to each other; but suppose the power P (Fig. 89.) to act upon the weight W by a string going over the movable pulley D in an oblique

Fig. 89.

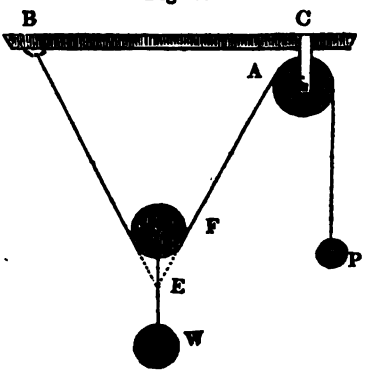
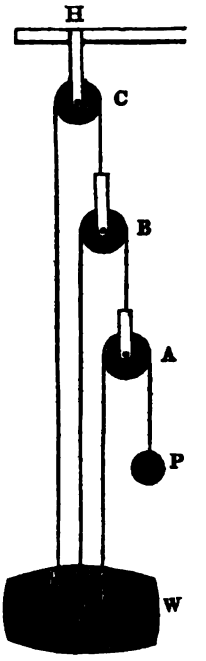


Fig. 90.



direction; then produce the string AF to E, and draw DF at right angles to DE, (D being the center of the pulley.) Let FE represent the magnitude of the power acting in the direction EF, which resolve into ED, DF; then ED is that part of it which is efficacious in supporting the weight W; and since the string BD supports the same weight as the string AF, the whole weight sustained by the string BFA will be represented by 2DE; hence,

P : W :: EF : 2DE :: rad. : 2 cos. DEF; or,

the power is to the weight, as radius to twice the cosine of the angle of inclination of the direction of the power to that of the weight.

125. There is another mode of combining pulleys together, which we have not yet noticed; viz. when each string is fixed into the weight, as in Fig. 90. In this case, supposing there is an equilib-

rium, and the power P acts upon a string going freely over the pulley A, then it is evident that the pressure upon that pulley will be equal to 2P, ∴ the string BA supports a part of the weight equal to 2P. For the same reason, since the string FBA goes freely over the pulley B, the string CB supports 4P, &c.; hence the portions of weight supported by the strings AG, BF, CE, &c. are P, 2P, 4P, &c. respectively, and consequently $W=P+2P+4P+$&c. $\ldots 2^{n-1}P$ (where $n=$ number of strings attached to the weight) $=P(1+2+4+$&c. $2^n-1)=P(2^n-1)$,* $\therefore P : W :: 1 : 2^n-1$. The pressure upon the support at H is evidently equal to $P+W=P+(2^n-1)P=2^nP$.

126. Examples.

1. *A weight of* 56 *pounds is kept in equilibrio by a power of* 7 *pounds by means of a system of pulleys, in which the same string goes round every pulley: What is the number of movable pulleys?*

Let $n=$ the number of strings at the lower block of pulleys; then, by Art. 122, $P(7) : W(56) :: 1 : n=\frac{56}{7}=8=$ twice the number of movable pulleys, ∴ the number required is 4.

2. *In the system of pulleys described in Art.* 123, *find the general relation between* P, W, *and* n.

In this system, $W=2^nP$, $\therefore P=\frac{W}{2^n}$;

$$\text{and } 2^n=\frac{W}{P},$$

$$\therefore n.\log.2=\log.W-\log.P,$$

$$\text{or } n=\frac{\log.W-\log.P}{\log.2}.$$

From which it appears, that if any two of the three quantities P, W, and n be given, the third may be found.

3. *Find the general relation between* P, W, *and* n, *in the system of pulleys described in Art.* 125; *and also the number of pulleys*

* The sum of a geometric series, whose term is 1, common ratio 2, and number of terms n, is 2^n-1. (Alg. 442.)

necessary for a power of 3 *pounds, to support a weight of* 381 *pounds.*

$$\text{Here } W=P(2^n-1),\ \therefore P=\frac{W}{2^n-1};$$

$$\text{also, } 2^n-1=\frac{W}{P},\ \therefore 2^n=\frac{W}{P}+1=\frac{W+P}{P};$$

$$\text{hence } n\times\log.2=\log.(W+P)-\log.P, \text{ or } n=\frac{\log.(W+P)-\log.P}{\log.2}.$$

$$\text{If } W=381, \text{ and } P=3; \text{ then } n=\frac{\log.384-\log.3}{\log.2}=\frac{2.107210}{0.301030}=7.$$

127. Questions on the Principles of the Pulley.

1. *By means of a system of pulleys, of which five are movable, and in which the same string goes round all the pulleys, what power will be necessary to sustain a weight of* 165 *lbs.?* Ans. 16½ lbs.

2. *A weight is sustained by a power attached to a rope going over one movable pulley,* (*as in Fig.* 89.) *the direction of the rope making an angle of* 60° *with a vertical line passing through the center of the pulley: What is the relation between* P *and* W?

Ans. P=W.

3. *A weight of* 240 *lbs. is sustained by a power equivalent to* 7½ *lbs. by means of the system of pulleys described in Art.* 123: *What is the number of pulleys?* Ans. 5 pulleys.

4. *What power will be necessary to sustain a weight of* 2387 *lbs. in a system of* 10 *pulleys, constructed according to Fig.* 90, *where the strings are all fastened to the weight?* Ans. 2½ lbs.

CHAPTER VIII.

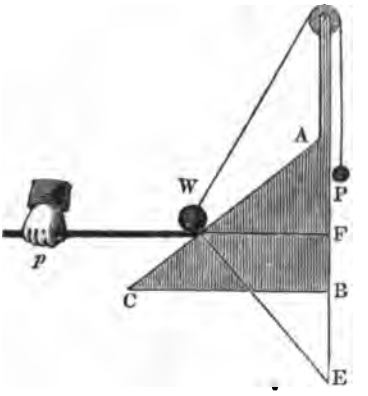
OF THE INCLINED PLANE, THE SCREW, AND THE WEDGE.

128. This chapter will comprehend the three remaining mechanical powers; viz. the *Inclined Plane*, the *Screw*, and the *Wedge;* beginning with the *Inclined Plane*, being that upon which the properties of the *Screw* more immediately depend.

THE INCLINED PLANE.

129. *In the inclined plane, an equilibrium is produced, when the power is to the weight, as the sine of the inclination of the plane is to the sine of the angle, which the direction of the power makes with a perpendicular to the plane, at the point where the weight rests upon it.*

Fig. 91.



Let AC (Fig. 91.) be an inclined plane, whose length is AC, height AB, and base BC; and suppose the weight W to be kept in equilibrio by any other weight (or power) P acting freely over a pulley fixed at D. Draw* WE at right angles to AC, meeting AB (produced if necessary) in E; the weight W may be considered as kept at rest by three forces, viz. the action of the power in the direction WD, its own weight (or gravity) in direction DE, and the reaction of the plane in direction EW; ∴ (by Art. 56.) these three forces are to each other as the three sides of the triangle DWE, in the directions of which they respectively act. Hence,

P : W : : WD : DE : : sin. WED or ACB : sin. DWE.

By the third law of motion, the pressure of W upon the plane must be equal to the reaction of the plane upon W; if, therefore,

* This figure is to be considered as a section of the plane passing through the center of gravity of the weight; and if the weight be not large, that center of gravity may be considered as placed in the angular point (W) of the triangle DWE.

EW represents that reaction, WE will represent the pressure upon the plane; hence,

P : press. on the plane : : WD : WE : : sin. WED or ACB : sin. WDE;
W : press. on the plane : : DE : WE : : sin. DWE : sin. WDE.

130. *In the inclined plane, when the power acts* PARALLEL TO THE PLANE, 1. *The power is to the weight, as the height of the plane to its length;* 2. *The power is to the pressure on the plane, as the height of the plane to its base;* 3. *The weight is to the pressure on the plane as the length of the plane to its base.*

If the power acts parallel to the plane, then WD may be considered as coinciding with WA, and the power, the weight, and the pressure, will be respectively represented by the three sides, WA, AE, WE of the triangle AWE; hence,

P : W : : WA : AE : : AB : AC : : height : length.
P : press. on the plane : : WA : WE : : AB : BC : : height : base.
W : press. on the plane : : AE : WE : : AC : BC : : length : base.

131. *In the inclined plane, when the power acts parallel to the* BASE *of the plane,* 1. *The power is to the weight, as the height of the plane to its base;* 2. *The power is to the pressure on the plane, as the height of the plane to its length;* 3. *The weight is to the pressure on the plane, as the base of the plane to its length.*

If the power acts parallel to the base of the plane, (i. e. if the weight W be sustained upon the plane by a force acting in the direction pW, and pushing upwards in such a manner that pW is always parallel to the horizon,) then produce pW to F; and when there is an equilibrium, the power, the weight, and the pressure will be respectively represented by the three sides WF, FE, WE of the triangle WFE; therefore,

P : W : : WF : FE : : AB : BC : : height : base.
P : press. on the plane : : WF : WE : : AB : AC : : height : length.
W : press. on the plane : : FE : WE : : BC : AC : : base : length.

132. *The least power will be required to raise or sustain a given weight upon a given inclined plane, when the direction in which that power acts is parallel to the plane; and, conversely, the greatest*

weight will also be raised or sustained by a given power upon a given inclined plane, when the direction in which the power acts is parallel to the plane.

Let α = angle of inclination of the plane, β = angle which the direction of the power makes with a perpendicular to the plane at the point where the weight rests upon it; then, by Art. 129, $P : W :: \sin.\alpha : \sin.\beta$; $\therefore P = \frac{W \times \sin.\alpha}{\sin.\beta}$. Suppose W and sin. α to be given, then P varies as $\frac{1}{\sin.\beta}$, and will consequently be the least when sin. β is the greatest, i. e. when the angle DWE becomes a right angle, or P acts parallel to the plane.

Again, $W = \frac{P \times \sin.\beta}{\sin.\alpha}$; if therefore, P and sin. α be given, then $W \propto \sin.\beta$, and will consequently be greatest when sin. β is greatest.

133. *The pressure on a given inclined plane, with a given power, is greatest when the power acts parallel to the base of the plane.*

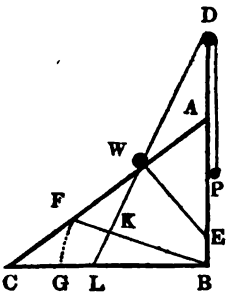
Let γ = WDE; then, by Art. 129, P : pressure upon the plane :: sin. α : sin. γ, $\therefore$ pressure upon the plane $= \frac{P \times \sin.\gamma}{\sin.\alpha}$; if P and sin. α be given, then the pressure upon the plane $\propto$ sin. γ, and will consequently be greatest when sin. γ is greatest, i. e. when WDE becomes a right angle, or the power acts parallel to the base of the plane.*

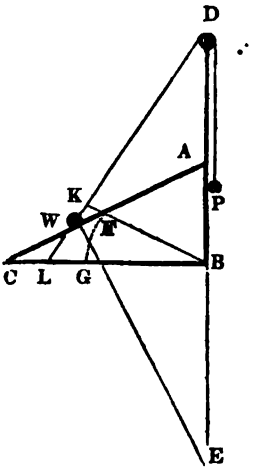
134. When a weight W is sustained by another weight P going over a fixed pulley D, (Fig. 92.) since the angle DWE varies at every point of the plane, it is evident that there is but one point of the plane where a given power will sustain a given weight; and that point may be thus determined. Take BG : BC :: P : W, and with center B and radius BG, describe a circular arc cutting AC

* In this case, since WE is greater than FE, it appears that the pressure is greater than W; but in this there is no inconsistency, for it is evident that when a weight is sustained upon an inclined plane by means of a force acting in direction pW, part of the pressure arises from the power as well as from the weight, and therefore the whole pressure may be greater than the weight.

in F; join BF, and draw DKL perpendicular to BF, or to BF produced; then the point W, where DL cuts AC, will be the point re-

Fig. 92.





quired; for the triangles DWE, BFC being similar,* we have

WD : DE : : BF or BG : BC; but

P : W : : BG : BC, ∴

P : W : : WD : DE; hence,

by Art. 129, the relation between the sides of the triangle WDE is such as to give the position of W when there is an equilibrium beween P and W.

But when the power acts parallel to the plane or parallel to the base of the plane, the ratio of P : W is constant† through every part of the plane; if therefore a given power be in equilibrio with a given weight at any one point of a plane, the same power would also be in equilibrio with the weight, when placed at any other point of that plane. We now proceed to give a few examples for illustration.

135. Examples.

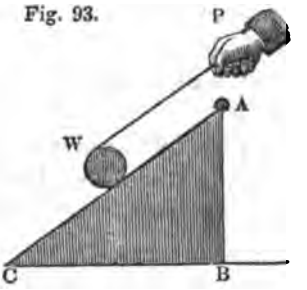
1. *A person is just able to sustain by his strength a weight of* 200 *pounds. What weight would he be able to sustain on an in-*

* Draw WE at right angles to AC, as before, then AEW=ACB; and in the right-angled triangles BKL, DBL, KLB is common, ∴ KBL=KDB; hence CFB =DWE, and consequently the triangle CFB is similar to the triangle DWE.

† For (in one case) P : W : : height of the plane : length of plane; and in the other, P : W : : height of plane : base of plane. (Art. 130, 131.)

clined plane whose elevation is 50°, *by means of a rope going round it, and fixed to the top of the plane in the manner represented in the annexed figure?*

Fig. 93.



In this case, (Fig. 93.) the power which supports the weight acts parallel to the plane, ∴ by Art. 130, P : W::AB : AC::sin. ACB (50°) : radius::7660 : 10000, or $W=\frac{P\times 10000}{7660}$; this is the weight supported by the rope PWA; but since that rope is fixed at A, each part PW, WA, of that rope supports half* the weight; hence, if the force exerted by the rope PWA = P, the force acting in the direction WP = ½P; calling that force (p), then $p=\frac{1}{2}P$, or $P=2p$; substitute this for P, and we have $W=\frac{2p\times 10000}{7660}=(\text{if } P=200)\frac{400000}{766}$ = 522.19 pounds; if therefore a person by his natural strength is able to lift a weight of 200 pounds, acting under the circumstances here represented, he will be able to sustain a weight of 522.19 pounds.

2. *Upon an inclined plane, whose length is* 20 *feet, and elevation* 30°, *a weight of* 3 *pounds is sustained by a power of* 2 *pounds, acting over a pulley fixed at the distance of* 10 *feet from the top of the plane, (in the manner represented in Fig.* 92.): *It is required to find the distance of* W *from the top of the plane, when there is an equilibrium.*

Since AC = 20 feet, and ACB = 30°, AB (=sin. 30°) will be equal to 10 feet, and consequently $BC=\sqrt{(AC^2-BA^2)}=\sqrt{300}$ = 17.32 feet; constructing the figure, therefore, as in Art. 134, we

* The case being similar to that of a weight supported by two parallel strings going over a pulley.

have BG : BC (17.32) : : P(2) : W(3), ∴ BG or BF $= \frac{2 \times 17.32}{3}$ =11.55 feet; hence, in the triangle BFC we have BC=17.32, BF=11.55, FCB=30°, from which the angles FBC, CFB are found to be respectively 18° 34′ and 131° 26′; but the triangle DWE is similar to the triangle BFC, ∴ WED=30°, WDE=18° 34′, and DWE=131° 26′.

Again, since DWE is 131° 26′, and AWE a right angle, DWA must be 41° 26′; hence, in the triangle DWA, we have AD=10 feet, WDA=18° 34′, and DWA=41° 26′, from which AW is found to be 4.812 feet, which gives the distance of W from the top of the plane when P is in equilibrio with W.

3. *A body is sustained upon an inclined plane, first by a power acting parallel to the plane, and afterwards by a power acting parallel to the base of the plane. Compare the pressures upon the plane in these two different cases.*

By Art. 130, when the power acts parallel to the plane,
W : press. on the plane (P) : : length : base.

By Art. 131, when the power acts parallel to the base,
Press. on the plane (p) : W : : length (L) : base (B). ∴
p : P : : L^2 : B^2.

Thus, in a plane whose elevation is 60°, (and whose length is consequently double* of its base,) it makes a difference of 4 : 1 as to the pressure upon the plane, whether a body is sustained upon it by a force acting parallel to the plane, or by one acting parallel to the base of the plane.

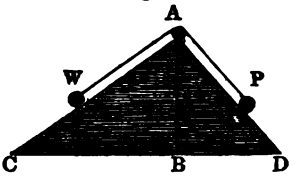
4. *Two weights* P, W, *resting upon the inclined planes* AC, AD, (*Fig.* 94.) *whose common height is* AB, *keep each other in equilibrio by means of a string going over a pulley fixed at* A. *Compare the two weights.*

Since the string passes freely over the pulley at A, and the two weights are at rest, it is evident that the tension of the string WAP must be every where the same, i. e. whatever power is exerted at A

* In Fig. 93, if ACB=60°, then CAB=30°; ∴ BC=½radius=½AC, or AC=2BC.

to sustain W on the plane AC, the same is exerted at that point to sustain P upon the plane AD; call that power (p), then, since the

Fig. 94.



power in each case may be considered as acting parallel to the plane, we have, by Art. 130.

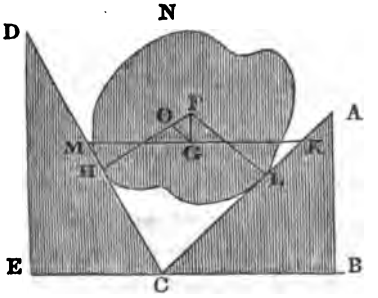
$$p : W :: AB : AC,$$

$$\text{and } P : p :: AD : AB;$$

$\therefore$ P : W :: AD : AC :: plane upon which P rests : plane upon which W rests. Hence, two weights resting on two inclined planes which meet, (forming a ridge,) will balance each other, when they are to one another as the lengths of the planes on which they respectively rest.

5. *A body is supported between two inclined planes of given elevations: Compare the pressure upon the planes.*

Fig. 95.



Let NHL, (Fig. 95.) represent a perpendicular section of the body passing through its two points of contact H, L with the planes; and from those points draw HF, LF at right angles to the planes DC, AC respectively. From their intersection F, draw FG perpendicular to the horizon, and let it represent the weight of the body. Through G draw GO parallel to LF; then the three sides of the triangle GOF will be in the direction of the three forces

which keep the body at rest upon the plane AC, viz. GO will represent the reaction of the plane AC; OF the reaction of the plane DC; and FG the weight of the body; and in the same manner it may be shown that the three sides of the same triangle will represent the three forces which keep the body at rest upon the plane DC. Through G draw MGK parallel to the horizon; then since the three sides of the triangle GOF are perpendicular to the three sides of the triangle MCK, MCK must be similar to GOF, (see note, p. 50.); and since the weight of the body, the pressure upon the plane DC, and the pressure upon the plane AC, are respectively represented by the three sides FG, FO, OG of the triangle GOF, they will also be represented by the three sides MK, MC, CK, of the triangle MCK. Hence,

Press. on DC : weight : : CM : MK : : sin. MKC or ACB : sin. MCK.
Weight : press. on AC : : MK : CK : : sin. MCK : sin. CMK or DCE.
Press. on DC : press. on AC : : sin. ACB : sin. DCE.

Thus suppose DCE=60°, ACB=30°, then pressure on DC : pressure on AC : : sin. 30° : sin. 60° : : $\frac{1}{2}$: $\frac{\sqrt{3}}{2}$: : 1 : $\sqrt{3}$. Hence, when a weight is supported between two inclined planes, the pressures on the planes are reciprocally as the sines of the angles of inclination of the planes.

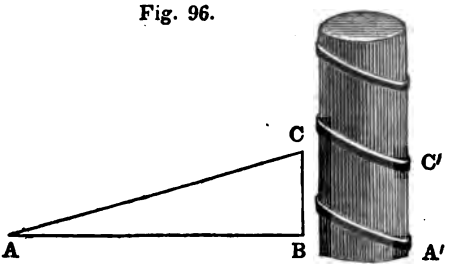
THE SCREW.

136. *The screw is a spiral thread or groove, winding round a cylinder, so as to cut all the lines drawn on its surface parallel to its axis, at the same angle. The spiral may be either on the convex or the concave surface of the cylinder, and the screw is denominated accordingly, the external or the internal screw.*

The distance between the two contiguous threads of a screw, corresponds to the height of an inclined plane, and the circumference of the cylinder corresponds to the base of the same plane; hence the forces necessary to produce an equilibrium in the screw, are the same as in the inclined plane. Thus let the inclined plane ABC (Fig. 96.) be wrapped round a cylinder, the circumference of whose base is equal to the line AB; then the point A being placed

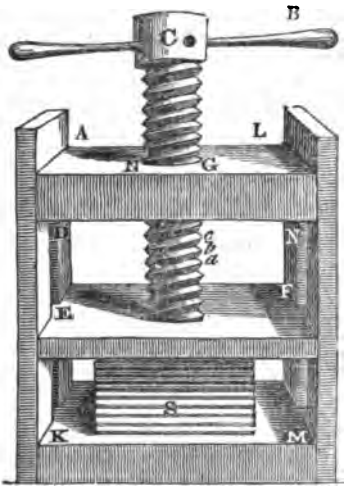
on A′, the point B will come round to A′, and the point C will fall on C′, and the line AC will trace out the thread of the screw on the surface of the cylinder as far as C′, and may be continued in the

Fig. 96.



same manner. By Art. 131, when the power acts parallel to the base of the plane, an equilibrium is produced when the power is to the weight as the height of the plane to its base; or, applied to the screw, an equilibrium is produced, when the power is to the weight, as the distance between two contiguous threads is to the circumference of the base.

Fig. 97.



187. Let the external and internal screw be fitted to each other in the manner represented in Fig. 97, and let the external screw be turned round by a power applied to the lever BC (acting parallel

to the base of the cylinder,) whilst the internal screw remains fixed; then it is evident, from the manner in which the two screws act upon each other, that whilst the lever BC makes one revolution, the external screw will be elevated or depressed through one of the spaces *ab*, *bc*, according to the direction in which it is turned. When the screw is depressed, it drives before it the board EF, which moves in the grooves DE, NF, by which means a pressure is created upon any substance (S) placed between that board and the fixed board KM.

138. Let us now estimate the quantity of this pressure, by finding the relation which it bears to the power which produces it. To do this, it will be necessary, in the first place to consider the force which would be generated in the elevation of the screw; which force may be estimated by showing separately what part of it arises from the action of the spirals of the screw upon each other, and what from the action of the lever. As the machine turns round, each point of the external screw acts upon the corresponding one of the internal screw with a force analogous to that by which a body is sustained upon an inclined plane when the power acts parallel to the base of the plane;* the whole force therefore of the screw will be of the same kind, and (by Art. 131.) will bear to the weight which it could support, the ratio of the distance between two spirals to the circumference of the cylinder. This would be the case, supposing the force to act close to the surface of the cylinder; when it acts therefore at the extremity of the lever BC, it will be increased in the proportion of the length of the lever to radius of the cylinder.

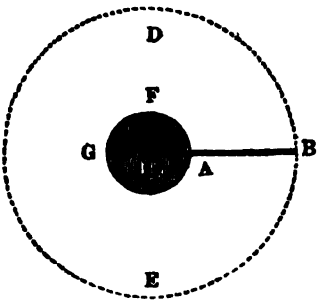
139. *In the screw, an equilibrium is produced when the power is to the weight, as the distance between two contiguous threads is to the circumference of the circle described by one revolution of the power.*

Let d=the distance between two spirals of the screw, which is cut upon the cylinder of which AFG is the section, a=length of

* Instead of moving a body up an inclined plane, we here move the plane itself against a resistance, which is overcome in the same manner as that of a body, and which may therefore be properly considered as a weight.

the lever (CB), π=3.1415, &c., then the circumference (BDE) of the circle described by the extremity of the lever $=2\pi a$. Let P= the power acting at the extremity of the lever, p= the power acting at the surface of the cylinder, W=the weight which is kept in equilibrio by P, then, by Art. 138,

Fig. 98.



p : W : : d : circumference AFG,

P : p : : CA : CB : : circumf. AFG : circumf. BDE;

$\therefore$ P : W : : d : circumf. BDE $=(2\pi a)$.

$$\therefore 2\pi a\mathrm{P}=\mathrm{W}d;\ \therefore \mathrm{P}=\frac{\mathrm{W}d}{2\pi a},\ \mathrm{W}=\frac{2\pi a\mathrm{P}}{d},\ d=\frac{2\pi a\mathrm{P}}{\mathrm{W}},\ \text{and}\ a=\frac{\mathrm{W}d}{2\mathrm{P}\pi};$$

hence, if any three of the four quantities W, P, a, d, be given, the fourth may be found.*

140. We have thus estimated the magnitude of the weight which might be sustained by a given power applied to the elevation of the screw; but this machine is oftener used for the purpose of creating a pressure than for raising a weight; and it is evident, that whatever force is exerted by the screw to sustain a weight when it is turned in one direction, will also be exerted to create a pressure down-

* From this proportion it appears that the relation P : W depends entirely upon the distance between the spirals and the circumference which the power describes, whatever be the thickness of the cylinder upon which the screw is cut; and since $\mathrm{P}\left(=\frac{\mathrm{W}d}{2\pi a}\right)\propto\frac{\mathrm{W}d}{a}$, when W is given, $\mathrm{P}\propto\frac{d}{a}$; i. e. the power necessary to sustain a given weight varies directly as the distance between the spirals, and inversely as the length of the lever.

wards, when it is turned in an opposite direction. Whether, therefore, the screw be applied to raising a weight, or creating a pressure, the power necessary to sustain the weight or produce the pressure, will always bear to that weight or pressure the ratio of the distance between any two spirals of the screw to the circumference of the circle which the power describes.

141. Examples.

1. *A screw, the distance between whose spirals is one inch, is turned horizontally by a lever whose length is* 2 *feet, reckoning from the axis of the screw: What weight could be sustained or pressure produced by it, when a power of* 30 *pounds acts at the extremity of the lever?*

By Art. 139, $W=\frac{2\pi aP}{d}=\frac{2\times3.1415\times24\times30}{1}=4523.76$ pounds; i. e. a power of 30 pounds applied to a machine of this kind would be sufficient to sustain a weight, or create a pressure, equivalent to about 4523 pounds, or somewhat more than two tons.

2. *A person who could just lift a weight of* 60 *pounds found himself able, by means of a lever* 3 *feet long, acting as a handle to a screw, to sustain a ton weight: What was the distance between the spirals of the screw?*

By Art. 139, $d=\frac{2\pi aP}{W}=\frac{2\times3.1415\times3\times60}{2240}=.504$ feet= about 6 inches.

3. *In Fig.* 99, *the screw* AB, *which is turned by a power* P *acting upon the handle* PQ, *turns at the same time the wheel* C, *in such a manner as to cause it to draw up the weight* W, *by a rope wound round the axle* D: *this is called the endless screw, and it is required to find the ratio of* P : W.

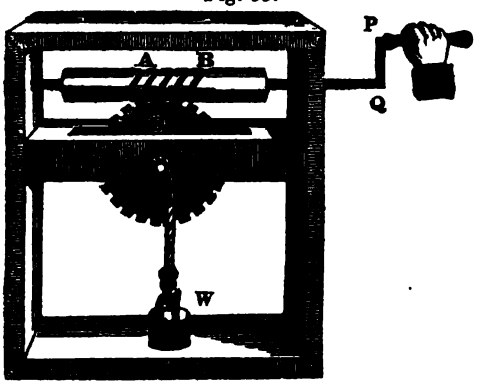
Supposing all the parts of this machine to be nicely adjusted, it is a very powerful one; inasmuch as it combines the energy of the screw with the multiplying power of the wheel and axle. To estimate the effect, therefore, let PQ$=a$, the distance between two spirals of the screw$=d$, radius of the wheel$=$R, radius of the axle

$=r$, $\pi=3.14159$, and Q=the force exerted by the screw upon the wheel; then, by Art. 139, $P : Q :: d : 2\pi a$,

and by Art. 113, $Q : W :: r : R$,

$\therefore P : W :: dr : 2\pi aR$,

Fig. 99.



Hence $P=\frac{Wdr}{2\pi aR}$, and $W=\frac{2P\pi aR}{dr}$; let $d=1$ inch, $a=12$ inches, $r=4$ inches, $R=18$ inches, and $P=30$ pounds, then $W=\frac{40715}{4}$ pounds $=4.54$ tons; so that, by means of this machine, a power of 30 pounds would be sufficient to keep in equilibrio a weight of about $4\frac{1}{2}$ tons.

142. Questions on the Principles of the Inclined Plane and Screw.

1. *If a man can draw a weight of* 125 *pounds up the side of a perpendicular wall,* 20 *feet high, what weight will he be able to raise along a smooth plank* 44 *feet long, laid sloping from the top of the wall?* Ans. 275 pounds.

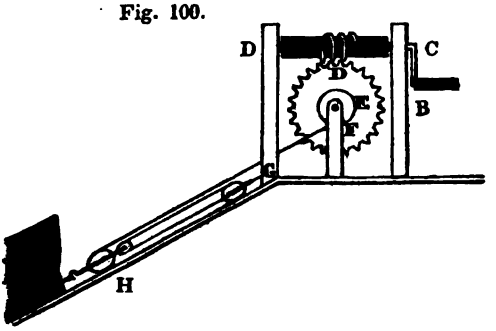
2. *Suppose that a horse is able to draw a weight of* 440 *pounds out of a well, (by a rope passing over a fixed pulley, which allows the horse to draw in a horizontal direction;) what weight will the same animal draw up a rail-way having a slope of five degrees, no allowance being made for friction?* Ans. 5048.5 pounds.

3. *A lever five feet long is fixed at right angles in a screw whose threads are one inch asunder, so that the lever turns just once round in raising or depressing the screw one inch. If then this lever be urged by a force of* 65 *pounds, with what force will the screw press?*

Ans. 24504.35 pounds.

4. *A ship-wright wishing to haul a ship upon the stocks, employed a machine, combining the lever, the screw, the wheel and axle, the pulley, and the inclined plane, as represented in* Fig. 100.

Fig. 100.



The handle of the winch BC =18 inches.
The distance of the threads on CD=1 inch.
The radius of the wheel ED =2 feet.
The radius of the axle EF =6 inches.
G is a fixed, and A a movable pulley,
the number of strings being =4.
Inclination of the plane =30°.

Allowing a man to turn on the handle at B, *with a force equal to* 100 *pounds, how much force could he exert on the ship?*

Ans. 361911.168 pounds, or more than 161½tons.

THE WEDGE.

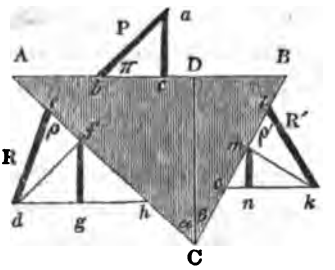
143. All those instruments which are used for the separation of the parts of bodies, such as knives, axes, coulters, and chisels, come under the general denomination of the wedge; but these instruments are made of such variety of shapes, and forces are applied to them in such various ways, that, of all the mechanical powers, the wedge is that whose properties are least capable of being brought to mathe-

matical calculation. In the particular case when the wedge is of the form of a triangular prism, and the resistance upon its sides can be considered as forces acting in given directions, the relation between those resistances and the power which counteracts them, may be estimated in the following manner.

144. *In the wedge, an equilibrium is produced, when that part of the power, which, when resolved, acts perpendicularly to the back of the wedge, is equal to the sum of those parts of the resistances which also act perpendicularly to the back; these resistances being to each other inversely as their respective distances from the line of direction in which the resultant of the power acts.*

Let ABC (Fig. 101.) represent a section of the wedge perpendicular to the axis of the prism, and suppose its sides AC, BC, to be perfectly smooth. Let a power P, whose magnitude and direction is represented by *ab*, act upon AB, the back of the wedge, and let it be counteracted by two resistances R, R′, (which are represented in quantity and direction by the lines *de*, *kl*,) acting upon

Fig. 101.

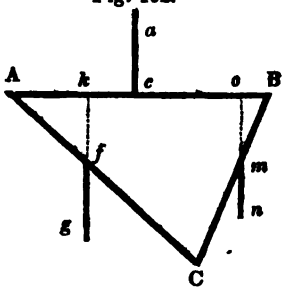


the sides AC, BC. Resolve *ab* into *ac* perpendicular, and *bc* parallel to the back of the wedge, and let *de* also be resolved into *df* perpendicular and *ef* parallel to the side AC; and since the side AC is perfectly smooth, *df* only is effectual to stop the progress of the wedge; resolve *df* again into *dg* parallel, and *gf* perpendicular to the back of the wedge, then *fg* is the only part of the resistance R which is directly opposed to that part of the power (viz. *ac*) which acts perpendicularly to the back of the wedge. Let the resistance R′ be resolved in the same manner, and let *mn* be that part of it which is directly opposed to *ac*; then, supposing every part of the wedge

to be perfectly hard and inflexible, it is evident that in the case of an equilibrium between P and R+R′, $ac=fg+mn$.

But in ascertaining this relation between P and R+R′ when they are in equilibrio, it should be recollected that the point c cannot be arbitrarily assumed; for there evidently will be a tendency to *vibratory* or *rotary* motion, unless the two resistances balance themselves

Fig. 102.



about that point. To determine the point c so that this tendency to vibratory motion shall be prevented, produce gf, nm (Fig. 102.) to k and o, then it is evident that the resistances fg, mn will only balance themselves about c, when, according to Theorem II. of parallel motion, (Art. 60.) $fg : mn :: co : ck$; that is, when the effective parts of the resistances are to each other inversely as their respective distances from the line of direction in which the resultant of the power acts.

145. But for the purpose of computing the actual power of the wedge in particular cases, we must find an expression for the power in known terms, arising from the shape of the wedge, and the conditions of the resistances. Therefore, produce dg to h, (Fig. 101.) and let a perpendicular CD, be drawn fron the vertical angle to the back of the wedge. Let α, β denote the two vertical angles, π the angle made by the power with the back of the wedge, and φ and φ' the respective angles made by the resistances R, R′ with the sides. Then since in the right-angled triangles dfh, CAD, the alternate angles dhf, CAD are equal, the angle fdg must be equal to the angle ACD(α).

Now $de(\mathrm{R}) : df ::$ rad. : sin. def, or sin. φ,

and $df : fg ::$ rad. : sin. fdg, or sin. α,

$$:: \mathrm{R} : fg :: \overline{\text{rad.}}|^2 : \text{sin. } \varphi \times \text{sin. } \alpha, \text{ or } fg = \frac{\mathrm{R} \times \text{sin. } \varphi \times \text{sin. } \alpha}{\overline{\text{rad.}}|^2}.$$

For the same reason, $$mn=\frac{R'\times\sin.\varphi'\times\sin.\beta}{\overline{\text{rad.}}|^2}.$$

Again, ab (P) : ac : : rad. : sin. π, $\therefore ac=\frac{P\times\sin.\pi}{\text{rad.}}$.

But when there is an equilibrium, $ac=fg+mn$; hence

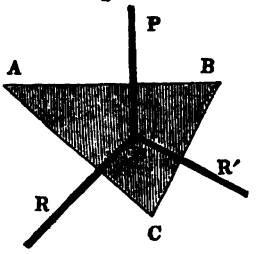
$$\frac{P\times\sin.\pi}{\text{rad.}}=\frac{R\times\sin.\varphi\times\sin.\alpha+R'\times\sin.\varphi'\times\sin.\beta}{\overline{\text{rad.}}|^2}.$$

$$\text{or } P=\frac{R\times\sin.\varphi\times\sin.\alpha+R'\times\sin.\varphi'\times\sin.\beta}{\text{rad.}\times\sin.\pi}.$$

This formula enables us to compute the power of a wedge, (or its ratio to the weight) when we have given the angles made by a perpendicular drawn from the vertical angle to the back of the wedge, and likewise the angles at which the power and resistances respectively act, whatever these angles may be; but when (as is frequently the case) the power acts perpendicularly to the back, and the resistances perpendicularly to the sides, the formula becomes much simpler. For then, sin. π, sin. φ, and sin. φ', each becomes equal to radius, and the general formula becomes $P=R\times\sin.\alpha+R'\times\sin.\beta$.

146. In the particular case when the directions of the power and resistances meet in the same point, as in Fig. 103, *the power is to the weight as the back of the wedge to the sum of the sides;* for then it is evident, that this equilibrium is produced under the same circumstan-

Fig. 103.



ces as that of a body kept *at rest* by three forces whose directions meet in that point; but (by Art. 57.) those three forces would be to each other as the three sides of a triangle perpendicular to their respective directions; P, R and R', will therefore be to each other as the three sides of the triangle ABC; i. e.

P : R : : AB : AC; P : R' : : AB : BC. $\therefore$

P : R+R' : : AB : AC+BC.*

* Algebra, Art. 388.

If the wedge be isosceles, and two equal resistances act perpendicularly to the sides, while the power acts perpendicularly to the back, then $P : W :: \frac{1}{2}AB : AC$;

For, $P : (2R)\ W :: AB : 2AC$. $\therefore$

$P : W :: \frac{1}{2}AB : AC$; that is,

If the wedge be *isosceles, the power is to the weight as half the back of the wedge is to one of its sides.*

Hence, the thinner the wedge, that is, the longer the sides of the wedge in proportion to its thickness, the greater is its power of overcoming resistance, with a given blow; a well known property of sharp instruments, which are referred to the wedge.

147. Examples.

1. *A wedge, whose sides are perfectly polished planes, is driven into the trunk of a tree, until that part of the pressure on each side, which acts perpendicularly to the back, is* 1500 *pounds: What force must be applied at the back to prevent its recoil?*

Here $P = R + R' = 2R = 3000$ pounds.

2. *Two equal resistances* (R, R′) *acting at angles of* 60° *and* 30° *upon the sides of a perfectly smooth wedge, are kept in equilibrio by a power acting perpendicularly to its back; the angle which a perpendicular (from the vertical angle of the wedge to the back) makes with the side upon which* R *acts is* 45°, *and with that upon which* R′ *acts* 30°: *Required the ratio of* $P : R + R'$ *or* $2R$.

This is a case of Art. 145, where $R' = R$, sin. π = rad. = 1, $\therefore P = R$, (sin. $\varrho \times$ sin. α + sin. $\varrho' \times$ sin. β); but,

$$\left.\begin{array}{l} \text{sin. } \varrho = \text{sin. } 60° = \frac{\sqrt{3}}{2} \\ \text{sin. } \alpha = \text{sin. } 45° = \frac{1}{\sqrt{2}} \\ \text{sin. } \varrho' = \text{sin. } 30° = \frac{1}{2} \\ \text{sin. } \beta = \text{sin. } 30° = \frac{1}{2} \end{array}\right\} \begin{array}{l} \therefore P = R\left(\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{4},\right) \\ \quad = \frac{R(\sqrt{6}+1)}{4}; \text{ hence} \\ P : R :: \sqrt{6}+1 : 4, \text{ and } P : 2R :: \sqrt{6}+1 : 8. \end{array}$$

148. Questions on the Principles of the Wedge.

1. *An isosceles wedge, whose acute angle was* 5°, *was inserted in a cleft of a rock. The pressure exerted perpendicularly on each side*

was equal to 6500 pounds: What force applied at the back of the wedge would overcome the resistance? Ans. 567.33 pounds.

2. *In a wedge of the form of a triangular prism, the power and resistances act perpendicularly to the back. The pressures upon the two sides of the wedge, whose lengths are 5 and 7 inches, are estimated at 950 and 300 pounds respectively, acting at the center of each side. The length of the back of the wedge is 5 inches: What power must be exerted perpendicularly to the back of the wedge, and to what point must it be directed, to produce an equilibrium?*

Ans. P=1250 pounds; and it must be directed towards the center of the back of the wedge. (Art. 144.)

3. *A force of 1600 pounds, acting at an angle of 75° upon the back of a wedge, is just sufficient to balance two resistances, which are to one another as 5 : 7, and whose directions make angles with their respective sides of 80° and 40°. The angle which a perpendicular from the vertical angle upon the back, makes with the side upon which is exerted the greater resistance, is 10°, and that which it makes with the other side is 4°: What is the amount of the resistances?* Ans. 16487.76 pounds.

General Principle, applicable to all the Mechanical Powers.

149. The most simple view which can be taken of the mechanical powers, is by a comparison of the respective VELOCITIES OF THE POWER AND THE WEIGHT. In order clearly to understand this subject, it must be recollected,

That an equilibrium implies the action of opposite and equal FORCES;

That the measure of a force is its MOMENTUM, *and, consequently, that in an equilibrium, the momenta on the opposite sides are equal;*

That momentum is compounded of the quantity of matter and velocity; and, hence, that a small body may have as much momentum as a large one, if it moves over as much greater space in the same time, as its quantity of matter is less.

Now let us apply the foregoing principles to the mechanical powers.

When the power and weight are in equilibrio, one has just as much momentum as the other; and therefore the product of the weight into its velocity, equals the product of the power into its velocity;

and it will be seen by reviewing the several mechanical powers, that in the theorem by which the law of equilibrium is in each case enunciated, the line into which the power or the weight is multiplied, is proportional to the space over which it moves in a given time, and therefore (by Art. 12.) is the measure of its velocity. This doctrine will be clearly comprehended by reviewing each of the mechanical powers separately.*

150. In the *straight lever*, (Fig. 90.) since the arcs Pp, Ww are described in the same time by P and W respectively, the velocity of P : velocity of W :: Pp : Ww :: Aa : Bb :: CA : CB :: W : P when P and W are in equilibrio.

In the *wheel and axle*, while the power descends through a space equal to the circumference of the wheel, the weight ascends through a space equal to the circumference of the axle, ∴ the velocity of the power : the velocity of the weight :: the circumference of the wheel : the circumference of the axle :: radius of the wheel : radius of the axle :: W : P.

In the single *fixed pulley*, the power and weight move through equal spaces in the same time, ∴ the velocity of the power is equal to the velocity of the weight; and in this machine P=W, ∴ velocity of power : velocity of weight :: W : P.

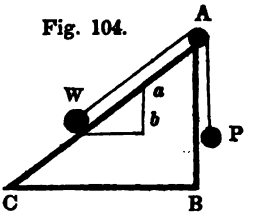
In the system of n pulleys, (Fig. 87.) whilst the power descends through any space (x), each of the strings belonging to the block of pulleys to which the weight is appended is shortened by $\frac{1}{n}x$, ∴ the weight ascends through a space equal to $\frac{1}{n}x$ in the same time that P descends through the space x; hence the velocity of the power : the velocity of the weight :: x : $\frac{x}{n}$:: n : 1 :: W : P.

In the system of pulleys, (Fig. 88.) whilst P descends through any space (x), the pulley F is raised through a space $=\frac{1}{2}x$; the pulley D through a space $=\frac{1}{2}\times\frac{1}{2}x=\frac{1}{4}x$; the pulley B through a space $=\frac{1}{2}\times\frac{1}{4}x=\frac{1}{8}x$; &c. hence (if n= the number of pulleys) velocity of P : velocity of W :: x : $\frac{1}{2^n}\times x$:: 2^n : 1 :: W : P.

* In estimating the increase of the power necessary to put the machine in motion, the friction of its different parts should be taken into the account. If p=the sum of the impediments arising from friction, and P=the power which would keep the weight (W) in equilibrio, then it is evident, that, before the machine can be put in motion, the power actually employed must exceed P+p.

Let ABC (Fig. 104.) be an *inclined plane*, up which the weight W is drawn by a power P acting over a pulley at A, then P : W:: AB : AC. Draw W*b* parallel to CB, and *ab* parallel to AB; then while P descends through a space equal to W*a*, W ascends upon the plane through the same space; but the space actually described by W in this time in the direction of gravity, is *ba*; ∴ (when the velo-

Fig. 104.



city of the power and weight are estimated in the direction in which they respectively act) the velocity of P : the velocity of W:: W*a* : *ab*:: AC : AB:: W : P.

In the *screw*, while P describes the circumference of a circle whose radius is BC, (Fig. 98.) the weight is elevated or depressed through a space equal to the distance between two contiguous spirals (*d*), ∴ the velocity of P : the velocity of W:: circumference (DBE) : *d*:: W : P.

Finally, in the *wedge*, the power of overcoming the resistance is proportioned (Art. 146.) to the acuteness of the wedge; and the distance to which the parts are separated, that is, the space over which the weight moves, when compared with the space through which the power (namely the wedge itself in the direction of the power) moves, is constantly diminished as the acuteness of the wedge is increased.

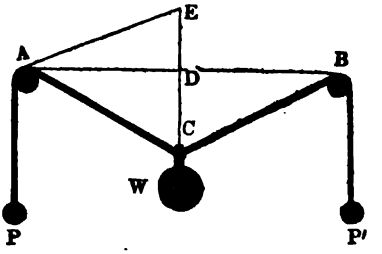
In the same manner we might trace this relation between the velocity of the power and the velocity of the weight, in machines where the power and weight act obliquely to each other; but as the operation then becomes somewhat more intricate, and as what has been already shown is sufficient to establish the truth of our proposition, it seems unnecessary to pursue the investigation any farther.

THE ROPE MACHINE.

151. *If a body fixed to two or more ropes, is sustained by powers which act by means of those ropes, this assemblage is called the Funicular or Rope Machine.*

152. *A given weight is in equilibrio with two given powers, which are equal to one another, and which pass over pulleys situated in the same horizontal line, when either power is to the weight, as the sine of the angle formed by the directions of the power and the weight, is to the sine of the angles formed by the directions of the two powers.*

Fig. 105.



Let A and B (Fig. 105.) be two pulleys fixed at a given distance from each other in the same horizontal line, and suppose a cord, PAWBP′, to pass over them, at the extremities of which are suspended the two equal weights P, P′; these two weights being kept in equilibrio by a third weight W.* Draw AE parallel to CB, and produce WC to meet it in E; then the three sides CA, AE, EC, of the triangle ACE, will represent the quantity and direction of the three forces by which the weight W is kept at rest; for these three forces are, 1st, The tension produced by P in direction CA; 2dly, the tension produced by P′ in direction CB or AE; and, 3dly, its own gravity in the direction EC; and since P′=P, AE or CB must be equal to AC. Hence P or P′ : W::AC or CB : EC::sin. AEC : sin. ACB.

* In this and the following articles we suppose the cords to be without weight, and perfectly inextensible; and the pulleys to be so small, and so adjusted, that the center of gravity of P, P′, W, &c. may all lie in the same vertical plane.

153. Join AB; then, since AB is parallel and CE perpendicular to the horizon, the angles at D are right angles, and since ACE and ACB are isosceles triangles, the lines AB, CE, bisect each other in D. Hence EC=2CD, and P : W::AC : (EC or) 2CD; $\therefore$ 2P : W :: 2AC :: 2CD::AC : CD, and $4P^2 : W^2 :: AC^2 : CD^2$; also, $4P^2 - W^2 : W^2 :: AC^2 - CD^2$ (AD^2) : CD^2 $\therefore$ $CD^2 = \frac{W^2 \times AD^2}{4P^2 - W^2}$, or $CD = \frac{W \times AD}{\sqrt{4P^2 - W^2}}$; which gives the position of W when the equilibrium takes place, for P, W, and AD ($\frac{1}{2}$AB) are known quantities.

Cor. 1. If $P = \frac{1}{2}W$, or $2P = W$, then $4P^2 = W^2$ and $4P^2 - W^2 = 0$; if W be greater than 2P, then $4P^2 - W^2$ is negative; in the former case, therefore, the value of CD becomes infinite, and in the latter impossible; which shows that if W be equal to or greater than 2P, no equilibrium can take place.

Cor. 2. If W and AD be given, CD will vary inversely as $\sqrt{4P^2 - W^2}$; and if P be indefinitely increased, W^2 may be neglected, and CD will vary inversely as $\sqrt{4P^2} = 2P$; that is, CD will diminish as the sum of the forces is increased, but can never become nothing until the sum of the forces becomes infinite. Suppose, for example, the line ACB to be a rope whose weight is equal to W; then *it would require an infinite force to draw it into a straight line, by powers applied at its extremities.*

CHAPTER IX.

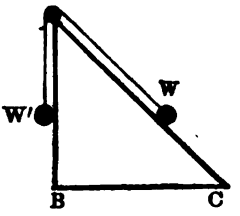
OF THE MOTION OF BODIES UPON INCLINED PLANES; AND THE DOCTRINE OF THE PENDULUM.

154. In the second chapter we investigated the relation which takes place between the space described, the velocity acquired, and the time of its motion, when a body ascends or descends perpendicularly near the earth's surface. The object of this chapter is, to ascertain that relation when the bodies ascend or descend upon planes inclined to the horizon.

MOTION ON INCLINED PLANES.

155. In Art. 130, it was shown, that if a weight W be sustained upon an inclined plane AC, (Fig. 106.) by another weight W′ acting upon it in a different direction parallel to the plane, then W′ : W::AB : AC. Suppose now the string WAW′ to be

Fig. 106.



cut in two, then it is evident that the weight W would descend down the plane with a force which bears to its own weight the ratio of AB : AC; and since (by Art. 134.) this force is constant through every part of the plane, the body thus descending may be considered as acted upon in every point of its descent by a constant force, which bears to the force of gravity the given ratio of the height of the plane H to the length of the plane L; call this force F, and let the force of gravity be represented by unity, then F : 1::H : L, and $F=\frac{H}{L}$;* i. e. the force which accelerates the motion of a body down an inclined plane is such a part of the force of gravity as may be represented by the fraction $\frac{H}{L}$; this force, therefore, differs not from the force of gravity in kind, but in degree; the effects produced by it must consequently be analogous to the effects produced by gravity.

156. In order to estimate these effects, we have only to consider, that if a body be acted upon by different constant forces for the same

* Since F : 1 :: AB : AC :: sin. ACB : rad. $\therefore F=\frac{\text{sin. ACB}}{\text{rad.}} \propto$ sin. ACB; therefore, the force which accelerates a body down an inclined plane varies as the sine of the angle of the plane's elevation.

time, the velocity generated will evidently be proportional to the intensity of those forces; and that if it be acted upon by the same force for different times, the velocity will be proportional to the time for which the forces act; from which it follows, that if a body be acted upon by different constant forces for different times, the whole velocity generated will be as the force and time conjointly.* Suppose now that the force of gravity is represented by unity, that $m = 16\frac{1}{12}$ feet, and that V = the velocity acquired in the time T whilst a body describes the space S acted upon by some other constant force F, then, from what has just been shown, V : the velocity acquired by gravity in $1'' = 2m$ (Art. 34.) $::F \times T : 1 \times 1$, $\therefore V = 2mFT$, and $T = \frac{V}{2mF}$. Again, since $V \propto F \times T$, $T \times V$ must vary as $F \times T^2$;† but $S \propto T \times V$ (Art. 29.) $\therefore S \propto F \times T^2$; hence S : the space described by gravity in $1''$ $(m) ::F \times T^2 : 1 \times 1^2$, $\therefore S = mFT^2$, and $T = \sqrt{\frac{S}{mF}}$. Lastly, since $V \propto F \times T$, V^2 must vary as $F \times T \times V$, or as $F \times S$,‡ $\therefore V^2$: square of the velocity acquired by gravity in $1''$ $(4m^2) ::F \times S : 1 \times$ space described by gravity in $1'' ::F \times S : 1 \times m$; hence $V^2 = 4mFS$, $V = 2\sqrt{mFS}$, and $S = \frac{V^2}{4mF}$.

157. Let us now apply these expressions to the case before us, i. e. let a body descend down an inclined plane whose height is (H) and length (L); then $F = \frac{H}{L}$ (Art. 155.); and if the body descends from rest,

Space (S) described in the time $T = \frac{H}{L} \times mT^2$, and $T = \sqrt{\frac{L \times S}{mH}}$.

Velocity (V) acquired in time $T = \frac{H}{L} \times 2mT$, and $T = \frac{L \times V}{2mH}$.

V_s acquired through space $(S) = 2\sqrt{\frac{H}{L} \times mS}$, and $S = \frac{L \times V^2}{4mH}$.

158. Since $S = \frac{H}{L} \times mT^2$, and H, L, m are given, $\therefore S \propto T^2$; that is, the space described varies as the square of the time when

* Algebra, 420. † Algebra, 413. ‡ Algebra, 418. cor.

a body falls from rest down an inclined plane, as well as when it descends freely by the force of gravity; the spaces described from rest in equal successive portions of time will therefore be as the odd numbers 1, 3, 5, 7, &c.; and if the body be projected upwards with the velocity acquired in falling through any space upon the plane, it will ascend to the point from which it fell, the spaces described in equal successive portions of time being as the numbers 1, 3, 5, 7, &c. taken in the inverted order. If, moreover, at any point of its descent, it moves forward with the velocity acquired continued uniformly, it will describe twice the space in the same time as that in which it has fallen to acquire the velocity; and if it be projected downwards or upwards with the velocity (V), and moves for the time (T), the space described in that time will be equal to $T \times V \pm \frac{H}{L} \times mT^2$. All this follows from the law of acceleration and retardation of bodies moving upon inclined planes, being the same as that which regulates the motion of bodies descending or ascending freely by the force of gravity. (See Arts. 31, 32, and 33.)

159. The expressions contained in Art. 157, apply to the case of a body descending from rest through any part of an inclined plane whose height is (H) and length (L). If the body falls through the whole length, then $S=L$, $\therefore$ the velocity acquired in falling down the whole length of the plane $\left(=2\sqrt{\frac{H}{L} \times mS}\right) = 2\sqrt{mH} =$ (by Art. 34.) the velocity acquired by descending freely through the height H; $T\left(=\sqrt{\frac{L \times S}{mH}}\right) = \frac{L}{\sqrt{mH}}$; but the time of falling freely down $H = \sqrt{\frac{H}{m}}$, $\therefore$ the time of describing the whole length of the plane : the time of falling freely down its height $:: \frac{L}{\sqrt{mH}} : \sqrt{\frac{H}{m}} :: L : H$:: length of the plane : height of the plane.

Since $T = \frac{L}{\sqrt{mH}}$, $V = 2\sqrt{mH}$, and m is a given quantity, T varies as $\frac{L}{\sqrt{H}}$, and V as $\sqrt{H}$; i. e. the time of describing any inclined

plane varies as its length directly, and as the square root of its height inversely; and the velocity acquired varies as the square root of the height, whatever be the length of the plane.

Hence we deduce the following THEOREMS.

I. *The* VELOCITY *acquired in falling down an inclined plane, is the same as that acquired by falling freely through the perpendicular height of the plane.*

II. *The* TIME *of describing the whole length of an inclined plane, is to the time of falling freely through its height, as the length of the plane to its height.*

III. *The time of describing any inclined plane* VARIES AS *its length directly, and as the square root of its height inversely.*

IV. *The velocity acquired in describing any inclined plane* VARIES AS *the square root of its height, whatever be the length of the plane.**

160. EXAMPLES.

1. *How far will a body descend from rest in* 4″, *upon an inclined plane whose length is* 400 *feet and height* 300 *feet?*

Ans. 193 feet.

2. *How long would a body be in falling down* 100 *feet of a plane, whose length is* 150 *feet and height* 60 *feet?* Ans. 3.94 seconds.

3. *The length of an inclined plane is* 60 *feet, and its elevation* 30°: *What velocity would a body acquire in falling from rest down it for* 2″? Ans. $32\frac{1}{2}$ feet in 1″.

4. *The height of a plane : length of a plane* :: 7 : 15: *How long would a body be in falling down it, to acquire a velocity of* 20 *feet per second?* Ans. 1.33 seconds.

5. H : L :: 5 : 14: *What space must a body fall through, to acquire a velocity of* 10 *feet per second?* Ans. 4.35 feet.

6. H : L :: 25 : 90: *What velocity would a body acquire in falling down* 70 *feet?* Ans. 35.37 feet in 1″.

* The expressions deduced in this section are true only when the body slides down a perfectly smooth plane; for in this case it is evident that every particle of the body is equally accelerated, and therefore whatever is proved of any one point of it will apply equally to all; but if the body in its fall has a rotary motion communicated to it, then it is evident that all the points of it will not be equally accelerated.

7. *The length of an inclined plane is* 100 *feet, and its elevation* 60°: *How long would a body be in falling down it, and what velocity would it acquire at the end of its fall?*

Ans. T=2.68 seconds; V=74.64 feet in 1″.

8. *A body is projected up an inclined plane whose length is* 10 *times its height, with a velocity of* 30 *feet in* 1″: *In what time will its velocity be destroyed?*

Ans. The time in which a body would fall down an inclined plane of this elevation to acquire a velocity of 30 feet per second =9.32″.

9. *A body is projected up an inclined plane, whose height is* $\frac{1}{4}$ *of its length, with a velocity of* 50 *feet per second: Find its place, and velocity after* 6″ *are elapsed?*

Ans. S=203$\frac{1}{2}$ from the bottom of the plane; V=17$\frac{1}{2}$ feet in 1″.

10. *A body falls from rest down the inclined plane* AC (*Fig.* 107.): *Compare the times of describing the first and last halves of it.*

Bisect AC in D, and draw DE parallel to CB; by Art. 159, the time down AC : the time down AD :: $\frac{AC}{\sqrt{AB}} : \frac{AD}{\sqrt{AE}}$:: (by similar triangles) $\frac{AC}{\sqrt{AC}} : \frac{AD}{\sqrt{AD}}$:: $\sqrt{AC} : \sqrt{AD}$:: $\sqrt{2} : 1$. Hence,

Fig. 107.

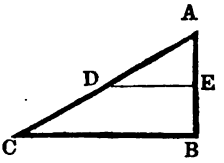
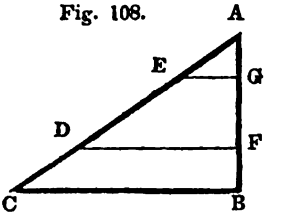


Fig. 108.



(1.) The times down different parts of the same inclined plane (when the body falls from rest from the top of the plane) are to each other as the square roots of the lengths of those parts.

(2.) The time down AC − the time down AD (i. e. the time down DC) : the time down AD :: $\sqrt{2}-1 : 1$.

11. *To mark out upon a plane* AC (*Fig.* 108.) *a part* ED *which shall be equal to the height* AB, *and which a body* (*falling down* AC) *would describe in the same time as one falling freely through* AB.

Let $AC=a$
AB or $ED=b$
$AE=x$,
then $AD=b+x$

$T\!AD$* : $T\!AE$:: $\sqrt{AD}$: $\sqrt{AE}$
:: $\sqrt{b+x}$: $\sqrt{x}$,

$\therefore T\!AD - T\!AE$ (i. e. $T\!DE$) : $T\!AE$:: $\sqrt{b+x}-\sqrt{x}$: $\sqrt{x}$,

but $T\!AE$: $T\!AC$:: $\sqrt{AE}$: $\sqrt{AC}$:: $\sqrt{x}$: $\sqrt{a}$,

and by Art. 159. $T\!AC$: $T\!AB$:: a : b :: $\sqrt{a}$: $\frac{b}{\sqrt{a}}$;

† $T\!DE$: $T\!AB$:: $\sqrt{b+x}-\sqrt{x}$: $\frac{b}{\sqrt{a}}$.

But $T\!DE = T\!AB$, $\therefore \sqrt{b+x}-\sqrt{x}=\frac{b}{\sqrt{a}}$;

which equation solved gives $x=\left|\frac{a-b}{4a}\right|^2$ or $AE=\left|\frac{AC-AB}{4AC}\right|^2$

161. Questions on the Inclined Plane.

1. *The length of an inclined plane is* 480 *feet, and the height* 210; *a body falls from rest from the top of the plane: What space will it have fallen through in* 6″; *what time will it be in falling through* 450½ *feet; and what velocity will it have acquired, when it has arrived within* 124.7 *feet of the bottom of the plane?*

Ans. $S=253\frac{5}{16}$ feet; $T=8$ seconds; $V=100$ feet in 1″.

2. *A body has been falling for* 15″ *down an inclined plane whose length is* 2½ *times its height: What velocity will it have acquired at the end of its fall?* Ans. $V=193$ feet in 1″.

3. *The elevation of a plane is* 30°; *a body in falling from the top to the bottom of it, acquires a velocity of* 579 *feet in* 1″: *What is the length of the plane?* Ans. $L=1042.2$ *feet.*

4. *A car broke loose from the top of an iron railway eleven miles in length, which was uniformly inclined to the horizon at an angle of* 1 *degree. Supposing the car to move without resistance, in what time would it reach the lower end of the railway, and what velocity would it acquire?* Ans. $T=7'\ 34''.88$; $V=127.68$ feet per. sec.

5. *At Alpnach, in Switzerland, is a celebrated slide for conveying timber trees from Mount Pilatus to Lake Luzerne, whence they are*

* $T\!AD$ means the time down AD, and so of the rest. † Alg. 393.

transported down the Rhine. The slide consists of an inclined plane formed of logs in the shape of a trough, into which the trees are launched, and down which they descend by the force of gravity. It is 8 *miles in length, and is inclined to the horizon, on an average, at an angle of* 3° 14′: *In what time will a tree descend from the top to the bottom of this plane, no allowance being made for friction?* Ans. 3′ 38″.

6. *Trees descending the slide sometimes "bolt out" of the trough, and occasion great destruction: With what force would a tree weighing* 1500 *pounds, leaping out of the slide at the end of* 7 *miles, strike upon an obstacle, as for example, upon a standing tree?*

Ans. With a force equal to 549318.2 lbs.

7. *Two inclined planes have a common height of* 75 *feet; the elevation of one of them is* 50°, *of the other* 20°: *With what velocity must a body be projected from the bottom of the former, that it may just rise to the top of the latter; and what will be the whole time of its ascending and descending through the two planes?**

Ans. V=69.46 feet in 1″; T=9.133 seconds.

8. *How long will a body be in falling down the last half of a plane, whose height is* 1 *mile, and angle of elevation* 1′?

Ans. T=5 hours, 4 minutes, 3 seconds.

9. *The length of a plane is* 250 *feet, and height* 150: *Mark out upon it a part equal to the height which a body in falling down it describes, while another body would descend freely through the height.*

Ans. It begins to describe this part when it has fallen through 10 feet from the top of the plane.

MOTION OF BODIES DOWN DIFFERENT SYSTEMS OF INCLINED PLANES.

162. It has already been shown, that when a body descends down an inclined plane whose length is L and height H, the velocity acquired varies as $\sqrt{H}$, and the time of description as $\frac{L}{\sqrt{H}}$; let us

* The planes are placed as in Fig. 94; a body is projected from D with a velocity just sufficient to carry it to A, and then falls from rest down the plane AC.

now apply these expressions to finding the relation between the times and velocities of bodies falling down different systems of inclined planes.

163. Let AC, AD, AE, (Fig. 109.) &c., be a system of inclined planes having the same height, AB; then since the velocities acquired by bodies falling down these planes are as $\sqrt{AB}$, and the times of description as $\frac{AC}{\sqrt{AB}}$, $\frac{AD}{\sqrt{AB}}$, $\frac{AE}{\sqrt{AB}}$, &c. (i. e. as AC, AD, AE, &c.) it is evident that bodies falling down a system of planes of this kind would acquire at the end of their fall the same velocity,* and that the times of descriptions would be as their respective lengths.

Fig. 109.

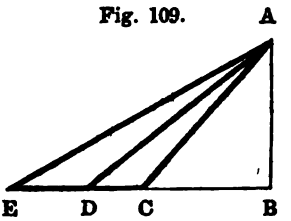
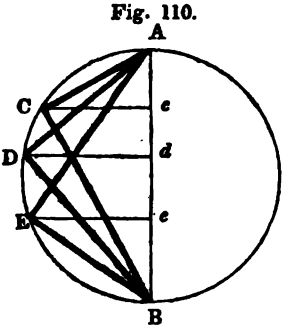


Fig. 110.



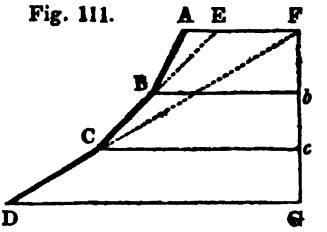
164. *If chords be drawn in a circle from the extremity of that diameter which is perpendicular to the horizon, the* VELOCITIES *which bodies acquire in falling through them, are proportional to their lengths; and the* TIMES *of describing these chords are all equal to one another, and are severally equal to the time of describing the diameter.*

Let the diameter AB of the circle ACB (Fig. 110.) be perpendicular to the horizon; draw the chords AC, AD, AE, and CB, DB, EB; draw also C*c*, D*d*, E*e*, &c., parallel to the horizon; then the velocities acquired by bodies falling down the former system of chords are as $\sqrt{Ac}$, $\sqrt{Ad}$, $\sqrt{Ae}$, and down the latter as $\sqrt{cB}$, $\sqrt{dB}$, $\sqrt{eB}$,

* And by Art. 159, this velocity is equal to the velocity which a body would acquire by falling freely through the height (AB) of the plane.

but $Ac=\frac{AC^2}{AB}$,* $Ad=\frac{AD^2}{AB}$; and $cB=\frac{CB^2}{AB}$, $dB=\frac{DB^2}{AB}$; hence $\sqrt{Ac}$, $\sqrt{Ad}$, $\sqrt{Ae}$, vary as AC, AD, AE, and $\sqrt{cB}$, $\sqrt{dB}$, $\sqrt{eB}$, vary as CB, DB, EB. The times of describing AC, AD, AE, and CB, DB, EB, are as $\frac{AC}{\sqrt{Ac}}$, $\frac{AD}{\sqrt{Ad}}$, $\frac{AE}{\sqrt{Ae}}$, and $\frac{CB}{\sqrt{cB}}$, $\frac{DB}{\sqrt{dB}}$, $\frac{EB}{\sqrt{eB}}$; but each of these quantities is equal to $\sqrt{AB}$.†

Fig. 111.



165. Suppose now that a body falls down a system of planes AB, BC, CD, (Fig. 111.) inclined to each other, and that no velocity is lost in falling from one plane to the other. Draw AF parallel to the horizon, and produce DC, CB to meet it in F, E; through B, C, draw B*b*, C*c*, parallel to AF, and let fall FG at right angles to the horizon. By Art. 163, the velocity down AB=velocity down EB =velocity down F*b*; ∴ the velocity down AB+BC=velocity down EB+BC (or EC) =velocity down FC=velocity down F*c*; and reasoning in the same manner, the velocity down AB+BC+CD=velocity down FD=velocity down FG; i. e. the whole velocity acquired by a body falling down a system of planes of this kind (supposing no velocity lost at the points B, C, &c.) is equal to the velocity which a body would acquire in falling freely through their joint height FG.

166. *When a body moves over a system of planes, the velocity lost in passing from any plane to the succeeding one, is to the velocity it had acquired at the bottom of the former plane, as the versed sine of the angle formed by the planes to radius.*

* For (Euc. 8. 6.) Ac : AC :: AC : AB, or $Ac=\frac{AC^2}{AB}$

† For since Ac : AC :: AC : AB, $\frac{AC^2}{Ac}=AB$, ∴ $\frac{AC}{\sqrt{Ac}}=\sqrt{AB}$; and so of the rest.

Let BF represent the velocity acquired in falling to B, and let BF be resolved into two parts, viz. FD perpendicular to the plane BC, and BD coincident with it. Now BD represents that part of the velocity with which the body sets out from B to fall down BC; and therefore the velocity lost at B=BF−BD. With the center B and radius BF describe the arc FI; then the velocity lost at B =DI=versed sine of the angle DBF. Therefore the velocity lost at B : the whole velocity at B::DI : BF::vers. sin. angles formed by the planes : radius.

Fig. 112.

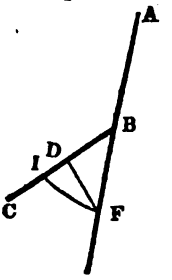
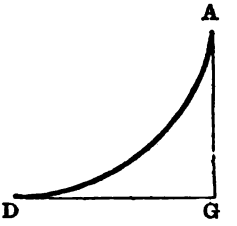


Fig. 113.



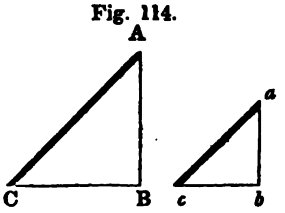
Suppose now the number of the planes to be increased, and their lengths and inclinations to each other to be diminished ad infinitum, then any two planes, AB, BC, will become two contiguous points in a curve, and consequently may be considered as coinciding with a tangent to that part of the curve; the angle DBF will therefore become equal to 0, and the velocities lost at the points B, C, &c., will also be equal to 0; hence, if a body falls down a perfectly smooth curvilinear surface AD, the velocity acquired will be equal to the velocity acquired in falling down its perpendicular height AG.

167. *The times and velocities of bodies falling down planes similarly inclined to the horizon, are to each other both as the square roots of the lengths and as the square roots of the heights of the planes.*

Let AC, *ac*, be two planes similarly inclined to the horizon, then AC : *ac*::AB : *ab*, and $\sqrt{AC} : \sqrt{ac} :: \sqrt{AB} : \sqrt{ab}$. Now the

time down AC : the time down $ac :: \frac{AC}{\sqrt{AB}} : \frac{ac}{\sqrt{ab}} :: \frac{AC}{\sqrt{AC}} : \frac{ac}{\sqrt{ac}} ::$

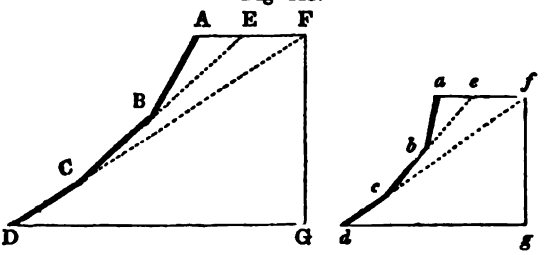
Fig. 114.



$\sqrt{AC} : \sqrt{ac}$, or $\sqrt{AB} : \sqrt{ab}$; and the velocity acquired in falling down AC : the velocity down $ac :: \sqrt{AB} : \sqrt{ab}$, or, $\sqrt{AC} : \sqrt{ac}$.

168. *The times of descent down* SIMILAR SYSTEMS *of inclined planes are as the square roots of the lengths of the planes.**

Fig. 115.



Let there be two systems of planes, AB, BC, CD, *ab*, *bc*, *cd*, similar, and similarly situated with respect to the horizon, and complete the figures as in Art. 164, then, supposing no velocity to be lost in passing from one plane to the other, and using the notation employed on page 145, we have, by Art. 167,

$$T\mathit{AB} : T\mathit{ab} :: \sqrt{AB} : \sqrt{ab}. \qquad \text{(X.)}$$

$$TEC : Tec :: \sqrt{EC} : \sqrt{ec} :: \sqrt{AB} : \sqrt{ab},\dagger$$

$$TEB : Teb :: \sqrt{EB} : \sqrt{eb} :: \sqrt{AB} : \sqrt{ab};$$

* The velocity down AB+BC+CD : velocity down $ab+bc+cd :: \sqrt{FG} : \sqrt{fg} :: \sqrt{FD} : \sqrt{fd} :: \sqrt{AB} : \sqrt{ab} :: \sqrt{(AB+BC+CD)} : \sqrt{(ab+bc+cd,)}$ the velocities as well as the times are therefore as the square roots of the lengths of the planes.

† For by similar triangles, EB : *eb* :: AB : *ab* ; and from similar planes, BC : *bc* :: AB : *ab* ; $\therefore$ EB + BC (or EC) : *eb* + *bc* (*ec*) :: AB : *ab*; and $\sqrt{EC} : \sqrt{ec} :: \sqrt{AB} : \sqrt{ab}$. In the same manner it may be shown that FD : *fd* :: AB : *ab*, or $\sqrt{FD} : \sqrt{fd} :: \sqrt{AB} : \sqrt{ab}$.

Hence,

$$T\overline{EC} - T\overline{EB}\ (\text{or } T\overline{BC}) : T\overline{ec} - T\overline{eb}\ (\text{or } T\overline{bc}) :: \sqrt{\overline{AB}} : \sqrt{\overline{ab}}. \quad (Y.)$$

In the same manner it may be shown, that

$$T\overline{CD} :: T\overline{cd} :: \sqrt{\overline{AB}} : \sqrt{\overline{ab}}. \quad (Z.)$$

From the proportions marked (X), (Y), (Z), therefore, we have

$$T(\overline{AB}+\overline{BC}+\overline{CD}) : T(\overline{ab}+\overline{bc}+\overline{cd}) :: \sqrt{\overline{AB}} : \sqrt{\overline{ab}}$$
$$:: \sqrt{(AB+BC+CD)} : \sqrt{(ab+bc+cd)}^*.$$

169. *The times of descending through* SIMILAR CURVES, *similarly situated with respect to the horizon, are as the square roots of the lengths of those curves.*

If the number of planes AB, BC, CD, &c. *ab*, *bc*, *cd*, &c., be increased, and their lengths and their inclinations to each other be diminished ad infinitum, then the polygons ABCD, *abcd*, become similar curves, in falling down which no velocity is lost.

Suppose the curves to be circular arcs; then, since similar circular arcs are to each other as the radii of the circles to which they belong, the times of descending through these arcs will be to each other as the square roots of their radii.

THE PENDULUM.

170. DEFINITIONS.—A *Pendulum* is a body suspended by a right line from any point, and moving freely about that point as a center.

The point about which the pendulum revolves, is called the center of suspension.

The vibration of a pendulum, is its motion from a state of rest at the highest point on one side, to the highest point on the other side.†

The center of oscillation of a pendulum, is such a point that, were all the matter of the pendulum collected in it, the quantity of motion

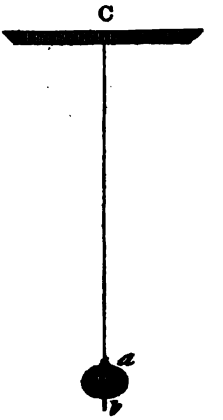
* For since $AB : ab :: BC : bc :: CD : cd$, $AB+BC+CD : ab+bc+cd :: AB : ab$; or $\sqrt{\overline{AB}} : \sqrt{\overline{ab}} :: \sqrt{(AB+BC+CD)} : \sqrt{(ab+bc+cd)}$.

† In these investigations, as in those of the Mechanical Powers, pendulums are supposed to move without any resistance from the air or from friction. The conclusions, therefore, will be accurately true only when applied to vibrations performed in a perfect vacuum, round a perfectly smooth axis of suspension.

(or momentum) would be equal to the sum of the momenta of all the parts taken separately.

Thus, the parts of the pendulum about *b* move faster than those about *a*, and consequently have more momentum; but there is a point about which the momenta balance each other, and therefore in the investigations relating to the pendulum, all the parts of which it consists may be considered as concentrated in that point.

Fig. 116.

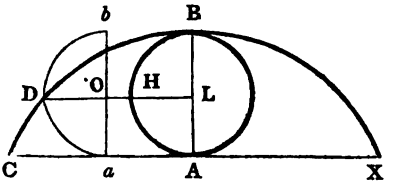


The center of oscillation is below the center of gravity; for, since the parts more remote from the center of suspension have more velocity than the parts that are nearer to it, the quantity of matter below the center of oscillation must be less than the quantity of matter above it.

171. In order to understand the doctrine of the pendulum, it is necessary to become acquainted with a few of the leading properties of the curve called the Cycloid.*

A Cycloid is the curve described by a point in the circumference of a circle rolling in a straight line on a plane.

Fig. 117.



Let the circle AHB make one revolution upon the line CAX, equal to its circumference; the curve line CDBX, traced out by

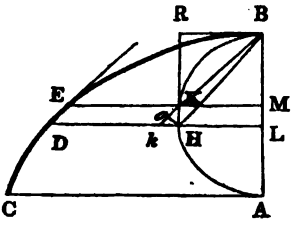
* The learner will remark that the mode of reasoning is this: it is first proved, that, were a pendulum to vibrate in a cycloid, all its vibrations, whether performed in larger or in smaller arcs, would be equal. It is then shown that pendulums vibrating in small *circular* arcs are subject, very nearly, to the same law. The Calculus affords an easier method than Geometry of investigating the properties of this as well as of other curves, but the geometrical method is retained in this treatise, to render the study intelligible to such as are not acquainted with the Calculus.

that point of the circle which was in contact with C when the circle began to revolve, is called a Cycloid. If CX be bisected in A, and AB be drawn at right angles to it, it is evident from the manner in which the curve is generated, that it will have similar branches on both sides of AB, and that its vertex B will be so placed as to make its axis AB equal to the diameter of the generating circle. Its properties, as applied to the vibration of the pendulum, are the following.

172. The *cycloidal ordinate* DH=*circular arc* BH.—For, let *b*D*a* (Fig. 117.) be the position of the circle when the generating point is at D; draw the diameter *ba* parallel to BA, and from D draw DHL parallel to CA; then the arc D*a*=arc HA, ∴ sin. DO= sin. HL, and consequently DH=OL; but from the mode in which the cycloid is generated, C*a*=arc D*a*, and CA=semicircle BHA; hence DH=OL=*a*A=CA−C*a*=semicircle BHA − arc HA= arc BH.

173. *A tangent to the cycloid at any point* E, *is parallel to the corresponding chord* BK *of the generating circle.*—Draw DHL indefinitely near to EKM; join BK, and produce it to *k*; let fall H*o* at right angles to K*k*. The indefinitely small triangle HK*k* is sim-

Fig. 118.

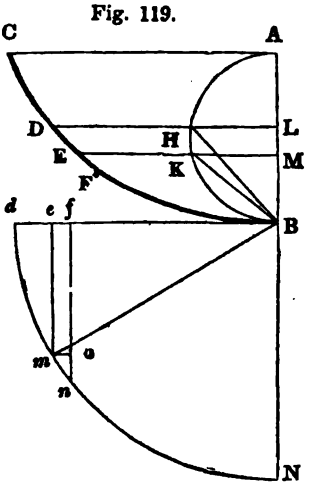


ilar to the triangle KRB formed by the tangents (KR, BR) to the circle at the points K, B, and is consequently isosceles; ∴ KH= H*k*. Now by Art. 172, arc BKH=DH, ∴ BKH − KH (=arc BK=EK) =DH−H*k*=D*k*; but since EK and D*k* are equal and parallel, ED and K*k* must also be equal and parallel; and as the tangent at the point E may be considered as coinciding with ED, it must therefore be parallel to the chord BK.

174. *The cycloidal arc* BE *is equal to twice the corresponding chord* BK *of the generating circle.*—For since the triangle KH*k* is isosceles, H*o* bisects the base K*k*, ∴ K*k* or ED = 2K*o*; and since H*o* may be considered as a small circular arc described with radius BH, K*o* = B*o* − BK = BH − BK; hence ED and K*o* are cotemporaneous increments of the cycloidal arc BE and the chord BK; and as the arc and chord begin together from the point B, and the former increases by ED or K*k* while the latter increases by K*o* = ½ED, the arc BE must be equal to twice the chord BK; consequently, the whole arc BC = twice the diameter AB.

175. AHB is a circle, whose diameter AB is perpendicular to the horizon, and CDB a cycloidal arc. Now, by Art. 159, if a body begins to descend from any point D, its velocity at the point E will be the same as the velocity at the point M of a body falling

Fig. 119.



freely through the perpendicular height LM; and its velocity at every other point during its descent through the cycloidal arc DEB will be the same as the velocity of the body falling freely at every other corresponding point of the line LMB. By Art. 167, therefore, the velocity V of the body thus descending along the cycloidal arc DE will vary as $\sqrt{(LM)} \propto \sqrt{(BL - BM)} \propto \sqrt{(AB \times BL -}$

$AB \times BM) \propto \sqrt{(HB^2 - KB^2)} \propto \sqrt{(DB^2 - EB^2)}$. Hence, let B*d* be drawn parallel to AC and equal to BD, and upon B*d* describe the quadrant of a circle *dm*N; take B*e* equal to BE, and draw *em* at right angles to B*d*; then will $em^2 = Bm^2 - Be^2 = Bd^2 - Be^2 = DB^2 - EB^2$, and consequently V, which varies as $\sqrt{(DB^2 - EB^2)}$, will vary as *em*, the sine of the arc *dm*, whose versed sine is *de* or DE, the space fallen through.

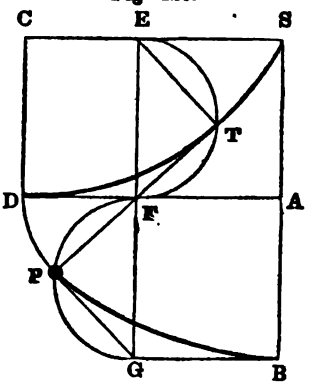
176. Let EF be an indefinitely small part of the cycloidal arc, and make *ef* equal to it; draw *fn* at right angles to B*d*, and *mo* parallel to it. Since EF is very small, it may be considered as described with the velocity V at E continued uniformly, and therefore the time of describing EF $\left(\text{since } T = \frac{S}{V}\right)$ will be represented by $\frac{EF}{V}$ or $\frac{ef}{em}$. Now since the sine *em* represents the velocity at any point E, the whole velocity acquired in falling down DB will be represented by the radius BN or B*m*; if, therefore, a body were to describe the quadrantal arc *dm*N with the velocity at the lowest point continued uniformly, the time of describing any small part (*mn*) of it would be represented by $\frac{mn}{Bm}$. But by similar triangles, B*me*, *mno*, $Bm : em :: mn : mo$ or ef, $\therefore \frac{ef}{em} = \frac{mn}{Bm}$; hence the time of describing the small cycloidal arc EF is equal to the time of a body moving through the corresponding small circular arc *mn* with the velocity in B continued uniformly; the whole time of descent therefore through DEB will be equal to the time of a body's describing the quadrantal arc *dm*N with the velocity at B continued uniformly.

177. Now the velocity at the lowest point B of the cycloid is equal to the velocity acquired in falling down the chord HB, and by Art. 31, with this velocity continued uniformly it would describe $2HB = BD = Bd$ in the same time. But by Art. 164, the time of falling down the chord HB = the time of falling down the diameter AB or the axis of the cycloid; hence the time of descending down the cycloidal arc DB, and the time of falling freely through the axis AB of the cycloid, are to each other as the times of describing the

arc *dm*N and the straight line B*d* with the same uniform velocity, i. e. as the quadrant of the circle to its radius.*

178. Take any line SC, and draw SA at right angles to it; make SC : SA : : semi-circumference of a circle : its diameter; and complete the parallelogram SCDA. Produce SA to B, making AB= SA; upon SC, AD, describe two semi-cycloids SD, DB, the vertex of the former of which is at D, and the latter at B; then if a body

Fig. 120.



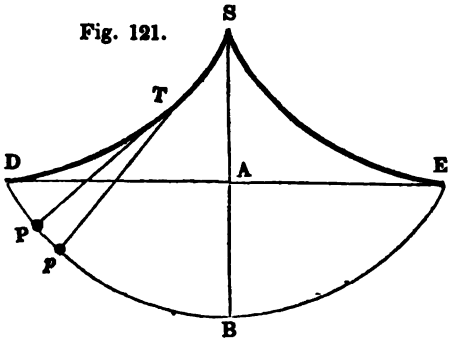
be suspended from a point S by a string whose length is equal to the cycloid SD, and begins to descend from D, its place will always be in the semi-cycloid DB, the part TP of the string being always at right angles to the cycloidal arc DB. For through any point F, draw EFG perpendicular to SC, and through B draw BG parallel to SC; then EG=SB; on EF, FG, describe the two semicircles ETF, FPG, and draw the chords TF, FP, the former of which (Art. 173.) is a tangent to the cycloid SD in T. Now SE=arc ET, and SC= ETF, ∴ CE(=DF)=arc TF=arc FP; hence the angles TEF, FGP are equal, and consequently the triangles TEF, FGP similar and equal to each other; ∴ TF, FP are in the same straight line; moreover, TP=2TF=(Art. 174.) the cycloidal arc TD; if therefore the string TP be always equal to the cycloidal arc TD, i. e. if the whole string STP be equal to the semi-cycloid SD, P will always

* In uniform motions, when the velocities are equal, the times are as the spaces. (Art. 12.)

be found in the cycloid DB; and since the angle TPG is a right angle, and PG (Art. 173.) is a tangent to the cycloid, therefore TP is always at right angles to the curve.

179. *A pendulum may be made to vibrate in a cycloid, by causing the flexible rod or string to apply itself, as it moves, to the sides of two semi-cycloids.*

Fig. 121.



Instead of supposing the body to descend down a plane in the form of a cycloid, the effect will evidently be the same with respect to the acceleration of the body, whether it be kept in a curve of this form by the reaction of the plane in a direction perpendicular to its surface, or by the tension of a string acting in the same direction. Now a small body P, suspended from the point S by a string STP of the same length with either of the semi-cycloids SD, SE, and made to vibrate between them, (the string gradually unwinding from the semi-cycloid SD as it descends to the lowest point B, and winding round the semi-cycloid SE as it ascends to the highest point E) will always be found in a cycloid similar and equal to the two semi-cycloids SD, SE; and as the string TP is always at right angles to the curve, it will be under precisely the same circumstances as the body descending down the plane. Having descended to the lowest point B, it will then ascend through an arc BE, equal to BD, in the same time in which it descended down DB; so that the whole time of one vibration will be twice the time of descending down DB; hence, (Art. 177.) *the time of one vibration will be to the time of a body's falling freely down half the length of the pendulum, as the circumference of a circle to its diameter.*

180. Since (Art. 179.) the time of a vibration is to the time down the axis in the same given ratio, whatever be the point from which the pendulum begins its oscillations, *all the vibrations of a pendulum of the same length are equal to each other.* But from the difficulty of constructing plates of the exact form SD, SE, and from other causes, the cycloidal pendulum is of little or no practical utility. In a geometrical point of view, however, this mode of comparing the time of a vibration with the time of falling down a space equal to $\frac{1}{4}$ the length of the string, is of considerable importance; for the cycloid at the lowest point B may evidently be considered as a circular arc described with the radius SB; if therefore a body be suspended by a string whose length is SB, and vibrates in a circular arc only to a very short distance on each side of the point B, *the time of the vibration of a pendulum in small* CIRCULAR *arcs, is to the time down half the length of the pendulum, as the circumference of a circle to its diameter, and therefore, within moderate limits, the time will be the same, whether the arc of vibration be larger or smaller.*

181. *The times of vibration of pendulums of different lengths are to each other as the square roots of the lengths.*

Let L=length of the string or thin inflexible rod by which a small body is suspended, $\pi=3.14159$, &c. $m=16\frac{1}{12}$ feet; then, (Art. 34.) the time of a body's falling down half the length of the string$=\left(\sqrt{\frac{S}{m}}=\right)\sqrt{\frac{L}{2m}}$. Hence the time of a vibration (T) : $\sqrt{\frac{L}{2m}}$:: π : 1, or $T=\sqrt{\frac{\pi^2 L}{2m}}$, which varies as $\sqrt{L}$. Let $T=1$, then $\sqrt{\frac{\pi^2 L}{2m}}=1$, or $\pi^2 L=2m$, and $L=\frac{2m}{\pi^2}=\frac{32.1666}{9.87}=3.259$ feet$=$ 39.11 inches; if the space fallen through from rest by gravity in 1″, therefore, be $16\frac{1}{12}$ feet, the length of a pendulum which vibrates seconds will be 39.11 inches.

Let $x=$ the space fallen through by gravity in 1″, then $T=\sqrt{\frac{\pi^2 L}{2x}}$ and $T^2=\frac{\pi^2 L}{2x}$, or $x=\frac{\pi^2 L}{2T}$; if therefore the length of a pendulum which vibrates in T″ be given, the space fallen through

by gravity in 1″ may be found; thus, let the length of the pendulum which vibrates seconds=39.2 inches, then $x=$ (for $T=1$) $\frac{\pi^2 L}{2}=\frac{9.87\times 39.2}{2}=193.452$ inches, or 16.121 feet in 1″.

182. *The times of vibration of pendulums of different lengths acted upon by* DIFFERENT ACCELERATIVE FORCES, *will vary as the square roots of the lengths directly, and as the square roots of the forces inversely.*

If the accelerative force be not given, then, (Art. 156.) the time of falling down ½ the length of the string $\left(\sqrt{\frac{S}{mF}}\right)=\sqrt{\frac{L}{2mF}}$, $\therefore T : \sqrt{\frac{L}{2mF}} :: \pi : 1$, or $T=\sqrt{\frac{\pi^2 L}{2mF}}$, which varies as $\sqrt{\frac{L}{F}}$.

The times of vibration of pendulums of the SAME LENGTH *vary inversely as the square roots of the accelerative forces.*

If L be given, then $T \propto \frac{1}{\sqrt{F}}$.

The lengths of pendulums vibrating in the SAME TIME *vary as the forces which accelerate them.*

And if T be given, then $L \propto F$.

183. *The* NUMBER *of vibrations performed in a given time by pendulums of different lengths, acted upon by different accelerative forces, are directly as the square roots of the forces, and inversely as the square roots of the lengths.*

Let n=the number of vibrations performed by a pendulum in any given time T', and T=the time of one vibration, then $T'=nT$, $\therefore n=\frac{T'}{T}$; but $T=\sqrt{\frac{\pi^2 L}{2mF}}$, $\therefore n=\sqrt{\frac{2mFT'^2}{\pi^2 L}}$, which varies as $\sqrt{\frac{F}{L}}$.

If therefore the lengths be given, the number of oscillations will be directly as the square roots of the forces; and if the forces be given, the number of oscillations will be inversely as the square roots of the lengths.

184. Examples.

1. *What is the time of an oscillation of a pendulum whose length is* 10 *feet; and what must be the length of a pendulum which shall oscillate ten times in a minute?*

The length of a pendulum which oscillates in 1″ is 39.11 inches; and, (Art. 181.) the times of vibration of pendulums of different lengths, are to each other as the square roots of their lengths; hence $1'' : T :: \sqrt{39.11} : \sqrt{120} :: 625 : 1095$, $T = \frac{1095}{625} = 1\frac{47}{125}$ seconds.

Again, let $T = 6'' =$ time of one oscillation of a pendulum which makes ten in a minute, then $6'' : 1'' :: \sqrt{L} : \sqrt{39.11}$, $\therefore 36 : 1 :: L : 39.11$, or $L = 39.11 \times 36$ inches $= 117\frac{1}{4}$ feet.

2. *Compare the times of vibration* T, *t, of two pendulums whose lengths are* L, *l, when carried to the distance* D, *d, above the Earth's surface.*

Let r = the radius of the Earth, then since the force of gravity varies inversely as the square of the distance of the Earth's center, the force which accelerates the pendulum whose length is (L) : the force which accelerates the pendulum whose length is $(l) :: \frac{1}{\overline{r+D}^2} : \frac{1}{\overline{r+d}^2} :: \overline{r+d}^2 : \overline{r+D}^2$; but by Art. 182, $T \propto \sqrt{\frac{L}{F}}$, $\therefore T : t :: \sqrt{\frac{L}{\overline{r+d}^2}} : \sqrt{\frac{l}{\overline{r+D}^2}} :: \overline{r+D}.\sqrt{L} : \overline{r+d}.\sqrt{l}$.

If $L = l$, then $T : t :: r+D : r+d$; i. e. *the times of vibration of the same pendulum, when carried to different heights above the Earth's surface, are to each other as the distances of those heights from its center.*

3. *If a pendulum at the Earth's surface vibrates* (m) *times in* T″; *how must its length be altered so that it may vibrate* (n) *times in* T″?

Let L = the length of the pendulum which vibrates (m) times in T″, and $L+x$ = the length of that which vibrates (n) times in T″; then by Art. 183, (since F is given) $m : n :: \frac{1}{\sqrt{L}} : \frac{1}{\sqrt{(L+x)}}$, $\therefore m^2 : n^2 :: L+x : L$, or $n^2L + n^2x = m^2L$, and $x = \frac{(m^2-n^2)L}{n^2}$.

Let $m=n\pm y$, where y is very small with respect to n; then $\frac{(m^2-n^2)L}{n^2}=\frac{(\pm 2ny+y^2)L}{n^2}=$ (rejecting y^2 as very small with respect to $2ny$) $\frac{\pm 2Ly}{n}$, and consequently $L+x=L\pm\frac{2Ly}{n}$, which furnishes us with a convenient theorem for ascertaining the quantity by which the pendulum of a clock must be lengthened or shortened according as it gains or loses a few seconds or minutes in a day.*

4. *A pendulum, which vibrated seconds and kept true time at the Earth's surface, was carried to the top of a mountain, and there lost* (t) *seconds in an hour: What was the height of the mountain?*

Let r=radius of the earth, x=height of the mountain, T=the number of seconds in an hour or the number of vibrations in an hour at the Earth's surface, then $T-t$ will be the number of vibrations in an hour at the top of the mountain. By Art. 183, when the length of the pendulum is given, the number of vibrations in a given time are directly as the square roots of the forces which act upon the pendulum; hence, since the forces are inversely as the squares of the distances from the center of the earth,

* Suppose, for instance, that a clock gains 3 minutes in a day, i.e. instead of performing $24\times60\times60$ or 86400 vibrations in a day, it performs $86400+180$ or 86580 vibrations in that time; then $n=86400$ and $y=180$, $\therefore \frac{2Ly}{n}=\frac{360L}{86400}=\frac{L}{240}$; and although this acceleration of the pendulum indicates that its length is a little less than 39.11 inches, yet it is evident that in finding the value of $\frac{L}{240}$, L may be assumed equal to 39.11 inches without any material error; hence $\frac{L}{240}$ or $\frac{39.11}{240}$ (=.16 of an inch) is the quantity by which the pendulum must be lengthened to make it vibrate seconds. In the example here given, m (=86580) is greater than n, and therefore y is positive; if the pendulum loses a certain number of seconds in a day, then m is less than n, and consequently y is negative; the value of $\frac{2Ly}{n}$ must in this case be subtracted from L, or the pendulum must be shortened. The theorem alluded to in the 3d example, therefore, gives this rule: Multiply twice the length of the pendulum by the number of seconds gained or lost, and divide the result by the number of seconds in a day; the quotient will give the number of inches or parts of an inch by which the pendulum is to be lengthened or shortened.

$$T : T-t :: \sqrt{\frac{1}{r^2}} : \sqrt{\frac{1}{r+x^2}} :: \frac{1}{r} : \frac{1}{r+x}, \therefore$$

$$t : T-t :: x : r, \therefore x = \frac{tr}{T-t}.*$$

185. Questions on the Pendulum.

1. *The lengths of pendulums vibrating seconds at St. Thomas near the equator, at New York, at London, and at Spitzbergen, in Lat. 80°, have been ascertained by very accurate experiments to be as stated below: Required the space through which a body would fall in one second at those places respectively?* (Art. 181.)

	Lengh of Pend.	Space in 1 sec.
St. Thomas,	39.02074	192.56 Inches.
New York,	39.10168	192.96
London,	39.13860	193.14
Spitzbergen,	39.21469	193.52†

2. *What is the length of the seconds pendulum at New Haven, where a body falls from rest* $16\frac{1}{12}$ *feet the first second?*

Ans. 39.110, or $39\frac{1}{12}$ inches nearly.

3. *The length of the seconds pendulum being* $39\frac{1}{12}$ *inches, what are the lengths of pendulums vibrating* $\frac{1}{2}$ *and* $\frac{1}{4}$ *seconds; also in 2 seconds; and how long must a pendulum be to vibrate once an hour?*

Ans. For half sec. 9.775 inch. $=\frac{1}{4}$ the length of the seconds pend.
quarter " 2.444 " $=\frac{1}{16}$ " "
two " 13.03 feet $=4$ times " "
one hour 7997.7 miles = diameter of the earth nearly.‡

* If t be very small with respect to T, then x may be considered as equal to $\frac{tr}{T}$ without any material error. Now $T = 3600''$, if therefore t be equal to 1, 2, 3, &c. seconds, then x willl be equal to $\frac{r}{3600}$, $\frac{2r}{3600}$, $\frac{3r}{3600}$, &c.; and supposing r to be equal to 4000 miles, the heights of mountains upon which a pendulum, that vibrates seconds at the Earth's surface, loses 1, 2, 3, &c. seconds in an hour, will be $1\frac{1}{9}$, $2\frac{2}{9}$, $3\frac{3}{9}$, &c. miles respectively.

† Hence it appears that the space through which a body falls from a state of rest in one second, is less at the equator than at the latitude of 80° by $\frac{25}{144}$ in. or nearly 1 inch.

‡ The diameter of the earth is 7912 miles.

4. A pendulum which vibrated seconds at the level of the sea, was found to vibrate but 3597 times in an hour, on the top of a neighboring mountain: Required the height of the mountain?

Ans. $3\frac{1}{4}$ miles.

CHAPTER X.

OF PROJECTILES.

186. *A body thrown into the air at any angle with the horizon, is called a projectile.*

The doctrine of Projectiles proceeds on the supposition that the force of gravity acts uniformly, and that bodies move without resistance from the air, neither of which suppositions is strictly true; but, for distances so small as those usually involved in these inquiries, the variation in the force of gravity is not material, and for all practical purposes the effect of the resistance of the air is separately computed, and allowed for.

187. *If a body be projected in any direction, not perpendicular to the horizon, it will describe a parabola.*

This proposition has already been demonstrated in Art. 48.

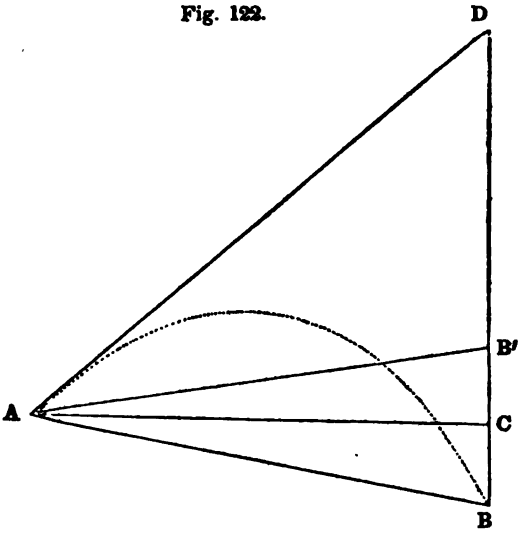
The most concise and comprehensive method of treating this subject, is first, to investigate by means of Analytical Trigonometry, a *general formula*, showing the relations between the time, velocity, range, and angle of elevation of a projectile, however these may vary among themselves; and secondly, to deduce from this, formulæ of greater simplicity that serve for particular cases.

188. In order to investigate the *general formula*,* let A (Fig. 122.) be the point of projection, AB or AB′ the plane on which the body is projected, passing through A. AB also denotes the *range*,

* See *Encyc. Metropolitana*, Art. Mechanics, p. 105, from which a large part of this chapter is taken.

or distance to which the body is thrown. Let AC be drawn parallel, and BCD perpendicular to the horizon; let the angle of elevation CAD$=a$, the angle of depression of the plane CAB$=b$, the velocity of projection $=v$, the time of flight $=t$, the range AB$=r$, and the space fallen through by gravity in one second or $16\frac{1}{12}$ feet $=m$.

Fig. 122.



Then, by the laws of uniform motion, at the end of the time t, if gravity did not act, the body would be found in the point D, while, by the laws of falling bodies, it would, in the same time, pass through the perpendicular DB; consequently,

$$AD=tv; \text{ and } DB=mt^2.$$

In the right-angled triangles ABC and ADC, the angle B is the complement of b, and the angle D is the complement of a; and, since the sides are as the sines of the opposite angles,

$$\cos.\ b : \sin.\ (a\pm b)^* :: tv : tv\,\frac{\sin.\ (a\pm b)}{\cos.\ b}=mt^2.$$

$$\text{Or, } \frac{mt}{v}=\frac{\sin.\ (a\pm b)}{\cos.\ b}. \qquad (1.)$$

$$\text{Again, } \cos.\ a : \sin.\ (a\pm b) :: r : \frac{r\sin.\ (a\pm b)}{\cos.\ a}=mt^2.$$

* *Plus* when the plane descends, *minus* when the plane ascends.

Or, $\frac{mt^2}{r} = \frac{\sin.(a \pm b)}{\cos. a}$. (2.)

From equation (1.) $t = \frac{v \sin.(a \pm b)}{m \cos. b}$.

and $t^2 = \frac{v^2 \sin.^2 (a \pm b)}{m^2 \cos.^2 b}$.

From equa. (2.) $t^2 = \frac{r \sin.(a \pm b)}{m \cos. a}$.

Hence, $\frac{r}{v^2} = \frac{\sin.(a \pm b) \cos. a}{m \cos.^2 b}$. (3.)

From these three equations, all the relations between the time, velocity, range, and angle of elevation, are readily determined; so that any two of these four quantities being given, the other two may be found. Thus,

By equation (1.) $v = \frac{mt \cos. b}{\sin.(a \pm b)}$.

By equation (2.) $r = \frac{mt^2 \cos. a}{\sin.(a \pm b)}$.

If the *range* and *elevation* be given to find the *time* and *velocity*,

By equation (2.) $t = \left(\frac{r \sin.(a \pm b)}{m \cos. a}\right)^{\frac{1}{2}}$.

By equation (3.) $v = \left(\frac{rm \cos.^2 b}{\sin.(a \pm b) \cos. a}\right)^{\frac{1}{2}}$.

If the *velocity* and *elevation* be given, to find the *time* and *range*,

By equation (1.) $t = \frac{v \sin.(a \pm b)}{m \cos. b}$.

By equation (3.) $r = \frac{v^2 \sin.(a \pm b) \cos. a}{m \cos.^2 b}$.

If any two of the above quantities be given to find the *angle of elevation*, then (the inclination of the plane to the horizon (b) being supposed to be known) in order to find the value of a, we must substitute for sin. $(a \pm b)$, its value involving the sine of each arc separately. Now sin. $(a \pm b)$ = sin. a cos. $b \pm$ sin. b cos. a*, whence either the sin. a or cos. a may be obtained.

The value of a and b being known, we shall then find, in all the preceding cases,

* Day's Trigonometrical Analysis, Art. 208.

Sin. $(a \pm b)$ cos. a equal to a known quantity. Let this be denoted by C; then

$$\text{cos. } a \text{ sin. } a \text{ cos. } b \pm \text{sin. } b \text{ cos.}^2 a = C;$$

Let $x = \text{sin. } a$, then $\text{cos. } a = \sqrt{(1-x^2)}$

$$x\sqrt{(1-x^2)} \pm \text{tan.}^* b\,(1-x^2) = \frac{C}{\text{cos. } b}.$$

189. As this is a quadratic equation, the solution will give two values of x, which shows, that there are always two different angles of elevation which equally satisfy the conditions of the problem, except in the case where the two roots of the equation are equal to each other, when only one angle will be found. In this case,

sin. $a = \text{sin. } \frac{1}{2}(90 \pm b)$, under which limitation the range will be a maximum, or the greatest possible; and all angles of elevation equally above and below that which gives the maximum range, will give ranges equal to each other. For, by the value of r as determined above,

$$r = v^2 \frac{\text{sin. } (a \pm b) \text{cos. } a}{m \text{ cos.}^2 b}.$$

If v and the angle b are given, the range will vary as sin. $(a \pm b)$ cos. a.

But sin. $(a \pm b)$ cos. $a = \frac{1}{2}$ sin. $(2a \pm b) + \frac{1}{2}$ sin. $(\pm b)$;† and since the second part of this expression is constant, the range will vary as

$$\text{sin. } (2a \pm b);$$

and this quantity will be greatest when

$$2a \pm b = 90°.$$

Then $$2(a \pm b) = 90° \mp b‡$$

And $$a \pm b = 45° \mp \tfrac{1}{2}b.$$

And since all angles equally above and below 90° have the same sine, all angles equally above and below that which gives the maximum have equal ranges. Thus, a cannon ball fired at an angle of 60° (in a vacuum) above a horizontal plane, would reach the plane at the same distance from the point of projection, as if fired at an elevation of 30°. When the data of the problem give or require a greater value for sin. $(2a \pm b)$ than 1, that is, than the sine of 90°, it shows the problem, under the proposed conditions, to be impossible.

* Day's Trigonometrical Analysis, Art. 216. † Ib. Art. 212. Eq. I.

‡ b being added to one side and subtracted from the other.

190. To find the greatest *height* to which the projectile will ascend, we must recollect that a body projected perpendicularly upwards will rise to the same height from which it must have fallen to acquire the velocity of projection. (Art. 30.) Calling the time of rising to the greatest height, t', we shall have $2mt'=$ the velocity of descent from gravity; and, v representing the whole velocity of projection in an oblique direction, the perpendicular part will be represented by $v \sin. a$; whence

$2mt'=v \sin. a$, and $t'=\frac{v \sin. a}{2m}$ and $t'^2=\frac{v^2 \sin.^2 a}{4m^2}$.

But the space fallen through in the time $t'=mt'^2=\frac{v^2 \sin.^2 a}{4m}$.

And the ascent in the same time from projection is,

$t'v \sin. a=\frac{v^2 \sin.^2 a}{2m}$. Consequently, the difference of these will be the greatest height of the projectile above the point A; that is,

$$h=\frac{v^2 \sin.^2 a}{2m}-\frac{v^2 \sin.^2 a}{4m}=\frac{v^2 \sin.^2 a}{4m}, \text{ and } \frac{4mh}{v^2}=\sin.^2 a. \qquad (4.)$$

If therefore the angle of elevation and velocity are given, the greatest height may be determined; or if the range (r) or the time (t) be given, (the angles being known,) the value of v^2 may first be ascertained by preceding formulæ, and then the height (h) from equation (4).

191. All the preceding equations become much more simple when the projection is along a *horizontal* plane; for then $b=0$, and $\sin. b=0$, and $\cos. b=1$; hence,

In eq. (1.) $\frac{mt}{v}\left(=\frac{(\sin. a \pm b)}{\cos. b.}\right)=\sin. a.$ (1′.)

(2.) $\frac{mt^2}{r}\left(=\frac{\sin.(a \pm b)}{\cos. a}\right)=\frac{\sin. a}{\cos. a}=\tan. a.$ (2′.)

(3.) $\frac{r}{v^2}\left(=\frac{\sin.(a \pm b)}{m \cos.^2 b.}\cos. a\right)=\frac{\sin. a \cos. a}{m}=\frac{\frac{1}{2}\sin. 2a}{m}.$ (3′.)

Hence from (1.′) $t=\frac{v \times \sin. a}{m} \propto v \times \sin. a.$ (5.)

(3′.) $r=\frac{v^2 \times \sin. 2a}{2m} \propto v^2 \times \sin. 2a.$ (6.)

(4.) $h=\frac{v^2 \times \sin.^2 a}{4m} \propto v^2 \times \sin.^2 a.$ (7.)

On a horizontal plane, therefore, (the most usual case,) we have the following THEOREMS.

I. *The* TIME OF FLIGHT *varies as the velocity of projection multiplied by the sine of the angle of elevation.*

II. *The* RANGE* *varies as the square of the velocity of projection, multiplied by the sine of twice the angle of elevation.*

III. *The* GREATEST HEIGHT *varies as the square of the velocity of projection, multiplied by the square of the sine of the angle of elevation.*

Moreover, since the sine of twice 45° equals the sine of 90°, which equals radius, hence, by Theorem II,

IV. *The* RANGE *is* GREATEST *when the angle of elevation is* 45°, *and is the same at elevations equally above and below* 45°.†

And since the square of the sine of the angle of elevation must be greatest when the angle is a right angle, therefore, by Theorem III,

V. *A projectile rises to the* GREATEST HEIGHT *when thrown perpendicularly upwards.*

Finally, since the sine of the angle of elevation is greatest when the angle is a right angle, therefore, by Theorem I,

VI. *The* TIME OF FLIGHT IS GREATEST, *when the body is thrown perpendicularly upwards.*

192. QUESTIONS ON PROJECTILES.

1. *A body is projected at an angle of* 15 *degrees with the horizon, with a velocity of* 140 *feet per second: What was its* range, greatest height, and time of flight?

By (6.) $r = \frac{v^2 \times \text{sin. } 2a}{2m} = \frac{19600 \times \frac{1}{2}}{32\frac{1}{6}} = 304.663$ feet.

By (4.) $h = \frac{v^2 \times \text{sin.}^2 a}{4m} =$ (By logarithms) 20.409 feet.

By (5.) $t = \frac{v \times \text{sin. } a}{m} = 2.253$ seconds.

* Sometimes called *random*.

† For the sine of twice any angle below 45° is the same as the sine of twice any angle the same number of degrees above 45°.

That is, its horizontal range was 304.663 feet; its greatest altitude, 20.409 feet; and its time of flight was 2.253 seconds.

2. *A body is projected at an angle of* 60° *with the horizon, and descends to it at the distance of* 100 *feet: With what velocity was it projected, and what was its greatest altitude and its time of flight?*

From (6.) $v = \left(\frac{2mr}{\sin. 2a}\right)^{\frac{1}{2}} = 60.94$ feet.

From (4.) $h = \frac{v^2 \times \sin.^2 a}{4m} = 43.3$ feet.

From (5.) $t = \frac{v \times \sin. a}{m} = 3.28$ seconds.

3. *I fired an arrow which remained in the air* 4 *seconds, and fell at the distance of* 100 *feet: With what angle of elevation was it fired, with what velocity, and how high did it ascend?*

By equation (2′.) $\tan. a = \frac{mt^2}{r} = 2.57333 = \tan. 68° 46'$.

By equation (6.) $v = \left(\frac{2mr}{\sin. 2a}\right)^{\frac{1}{2}} = 69.024$ feet per second.

By equation (4.) $h = \frac{v^2 \times \sin.^2 a}{4m} = 64.343$ feet.

4. *A gun was fired at an elevation of* 50°, *and the shot struck the ground at the distance of* 4898 *feet: With what velocity did it leave the gun, and how long was it in the air?*

Ans. Velocity, 400 feet per second.
Time, 19.05 seconds.

5. *Random* 4898 *feet, time of flight* 16 *seconds: Required the angle of elevation and the velocity of projection.*

Ans. El. 40° 3′, V. 400 feet per sec.

6. *Random* 2898 *feet, velocity of projection* 389.1 *feet: What were the elevation, and time of flight?* Ans. El. 19°, T. 7.87 sec.

7. *Elevation* 40°, *Random* 4898: *Required the random when the elevation is* $29\frac{1}{2}°$? Art. 191.(6.) Ans. 4263.

8. *Elevation* 40° 3′, *time of flight* 16 *seconds: Required the random and velocity of projection?* Ans. R. 4898, V. 400 feet.

9. *Velocity* 510 *feet per sec., time of flight* 15 *seconds, to find the elevation and random.* Ans. El. 28° 14′, R. 6740.

10. *On a side hill ascending uniformly above a horizontal level at an angle of* 10° 20′, *a ball was fired at an angle of elevation above the horizon of* 34°, *and with a velocity of* 401 *feet per second: What was the range on the side hill when the gun was directed up the hill, and what when directed downwards?*

Ans. 3438 and 5985 feet.*

CHAPTER XI.

OF THE STRENGTH OF MATERIALS.

193. The importance to the architect and the engineer of ascertaining the form and position of the materials which he employs, in order to secure the greatest degree of strength and stability at the least expense, has led mathematicians and writers on mechanics, to devote much attention to this subject. How is the strength of a beam affected by giving to it different shapes and different positions? how must a given quantity of matter be disposed in order that it may have the greatest possible strength? and upon what principles depends the stability of columns, roofs, and arches? these, and many similar inquiries, have been the objects of profound investigation.

The strength of *beams*, or pieces of timber, is the first object of inquiry. Strength is the power to *resist* fracture: stress, the power to *produce* fracture.

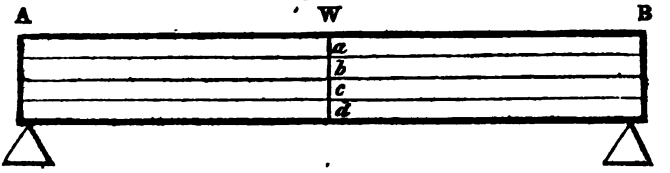
194. *The strength of a beam resting, horizontally, on its two ends, from a weight on its center, is proportioned to the area of a cross section, multiplied by the depth of its center of gravity.*

Thus, in Fig. 123, if AB represent a stick of timber, resting horizontally on supports at its two ends, and W be a weight placed at the center, and *a b c d* be a cross section, then, (supposing the weight to be sufficient to break the beam,) the fracture will commence at the bottom, and proceed regularly to the top, ending at

* See Art. 188, formula (3.)

W.* Now since the tendency to resist fracture from cohesion, must depend upon the total amount of all the separate forces acting be-

Fig. 123.



tween contiguous particles, it must evidently depend upon the number of the particles, that is, upon the extent of surface, or area of the section.

This would be the case were no other mechanical force involved but cohesion; but the *reaction* of the supports at A and B, being equal to the weight, and severally acting at the longer end of a bent lever AW*a*, AW*b*, &c., consequently, the tendency to fracture, from the leverage, will be lessened, as the shorter arm of the lever is increased, while the longer arm remains the same: therefore the strength being inversely as the stress, will be regularly increased as the distance of any lamina from W is increased, and the whole effect will be as the distance of the center of gravity from that point. Hence from both causes, the strength varies as the area of the section multiplied by the depth of the center of gravity.†

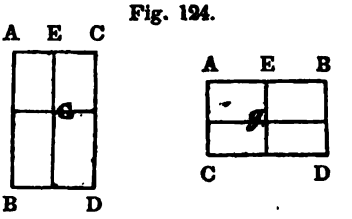
195. This proposition is general, and applies to a number of distinct cases. In *cylindrical* and *square* beams, since the area of the section varies as the square of its diameter, and the distance of the center of gravity from the point E varies as the diameter, their strength is as the *cubes* of the diameters. In beams of an *oblong* figure, the strength varies as the *breadth* and *square of the depth*; for here the area being as the product of the two sides, and the dis-

* It is here supposed, according to the views embraced by most writers on Mechanics since the time of Galileo, that the parts of a fractured beam turn about the line where the fracture terminates; but Mr. Barlow, in his Essay on the Strength and Stress of Timber, proves by experiment, that the tendency is to turn about a line entirely within the section, the fibers on that side of the line where the fracture begins being extended, and those on the other side compressed. This line he calls the *neutral axis*.

† Day's Algebra, (Art. 408.)

tance of the center of gravity from E being equal to half the perpendicular side, and therefore proportioned to that side, the proposition is, that the strength varies as the breadth × depth, × depth, or as the breadth into the square of the depth. Hence, the same oblong beam with its narrow side upwards, is as much stronger than with its broad side upwards, as the depth exceeds the breadth. For the area being the same in both cases, the strengths are proportioned to

Fig. 124.



EG and E*g*, or as AB to AC. Thus if a joist be ten inches broad and $2\frac{1}{2}$ thick, it will bear four times more weight when laid on its edge, than on its side. Hence the modern mode of flooring with very thin, but deep pieces of timber.

196. *In beams of different lengths, resting on two supports, the strength will vary as the area of the section into the depth of the center of gravity, divided by the length into the weight.*

Let L, *l*, denote the lengths; W, *w*, the weights; A, *a*, the areas of the sections; and G, *g*, the depths of the centers of gravity, of two prismatic beams, resting horizontally on their two ends.

The *stress*, or tendency to *produce* fracture, *from the weight of the beam itself*, will be expressed by $\frac{1}{8}L \times W$, and $\frac{1}{8}l \times w$.* But the tendency to *resist* fracture is denoted by $A \times G$, and $a \times g$. Hence the aggregate strength of the timber will be directly as the latter and inversely as the former. That is,

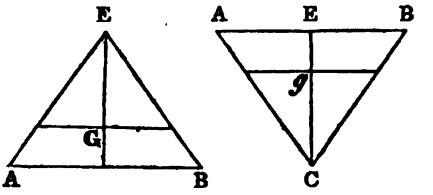
$$S : s :: \frac{A \times G}{\frac{1}{8}L \times W} : \frac{a \times g}{\frac{1}{8}l \times w} :: \frac{A \times G}{L \times W} : \frac{a \times g}{l \times w}.$$

197. *A triangular beam is twice as strong when resting on its broad base, as when resting on its edge.*

* For the reaction at each support is a force which $= \frac{1}{2}W$, acting upon $\frac{1}{2}L$ at its center of gravity; but the center of gravity of $\frac{1}{2}L$, is at a distance from the prop $= \frac{1}{4}$ of $\frac{1}{2}L = \frac{1}{4}L$; therefore the efficacy of the force $= \frac{1}{4}L \times \frac{1}{2}W = \frac{1}{8}L \times W$. (Fig. 123.)

For, the area being the same in both cases, the strength varies as EG and Eg, which are as 2 to 1. (Art. 71.)

Fig. 125.



These principles apply not only to beams, but to bars, and similar structures of every sort of matter.

198. *The strength of any bar in the direction of its length, is directly proportional to the area of its transverse section.*

Here each line of particles, in a longitudinal direction, may be considered as exerting a separate force, and therefore the aggregate force will manifestly depend on their number, and of course on the area of its section, the whole being equal to the sum of all its parts. Hence the various shapes of bars make no difference in their absolute strength, since this depends only on the area of the section, and must obviously be the same when the area is the same, whatever be its figure. A rope, therefore, or a wire, to which a weight is appended, is as likely to break in one place as in another; but when the weight of the rope becomes considerable, and the force is applied perpendicularly, the increase of weight, as its length increases, renders it more liable to break in the upper than in the lower parts.

199. *The lateral strengths of similar beams, are inversely as their lengths or breadths.*

Let D, d, represent the diameters of two *cylindrical* beams, or the sides of two beams in the form of *square prisms;* then,

$$S : s :: \frac{D^3}{L \times W} : \frac{d^3}{l \times w}.$$

But in similar beams, L $\propto$ D and W $\propto$ D^3 $\therefore$

$$S : s :: \frac{D^3}{D^4} : \frac{d^3}{d^4} :: \frac{1}{D} : \frac{1}{d} :: \frac{1}{L} : \frac{1}{l}.$$

In *oblong* beams,

$$S : s :: \frac{B \times D^2}{L \times W} : \frac{b \times d^2}{l \times w}.$$

But in similar beams, L $\propto$ B $\therefore$

$$S : s :: \frac{L \times D^2}{L \times D^2} : \frac{l \times d^2}{l \times d^2} :: \frac{1}{D} : \frac{1}{d} :: \frac{1}{L} : \frac{1}{l}.$$

When the beams are of the *same figure*, and their *lateral sections the same*, then, the breadth and depth of one being respectively equal to the breadth and depth of the other,

$S : s :: \frac{1}{L \times W} : \frac{1}{l \times w}$. And since $W \propto L$, $\therefore$

$S : s :: \frac{1}{L^2} : \frac{1}{l^2}$. Hence, half the length of a beam, supported at both ends, will bear four times as great a pressure as the whole beam; and a prop placed under the center of a beam increases its strength in the same ratio.

Long beams are weak from their own weight; and the length may be so increased, that they will break from this cause alone. The strength arising from making the beam larger, increases as the *square* of one of the homologous sides, while the weight increases as the *cube*, and therefore preponderates in long beams. To consider, in connection, the several circumstances which affect the strength of timber, it appears that a beam twice as broad as another of the same length, is also twice as strong; that one twice as deep, the other dimensions remaining the same, is four times as strong; and that one twice as long as another *similar* beam, has only half the strength.

200. But if besides their own weights, these beams be made to support other weights, W', w', placed at their middle points, then their tendency to fracture will be increased.

For, since the reaction, arising from the pressure on each support $= \frac{1}{2}W'$, and this force acts at the point of fracture with a leverage equal to $\frac{1}{2}L$, the stress produced by $W' = \frac{1}{2}L \times \frac{1}{2}W' = \frac{1}{4}L \times W'$. But the stress arising from the weight of the beam itself $= \frac{1}{2}L \times W$. Therefore the whole stress $= \frac{1}{2}L \times W + \frac{1}{4}L \times W' = \frac{1}{4}L(\frac{1}{2}W + W') \propto L(\frac{1}{2}W + W')$, $\therefore$

$$S : s :: \frac{A \times G}{L(\frac{1}{2}W + W')} : \frac{a \times g}{l(\frac{1}{2}w + w')};$$

or, in the case of cylinders and square prisms,

$$S : s :: \frac{D^3}{L(\frac{1}{2}W + W')} : \frac{d^3}{l(\frac{1}{2}w + w')}.$$

If the weights of the beams be so small, when compared with the weights supported, as to make it unnecessary to take them into consideration, then

$$S : s :: \frac{A \times G}{L \times W'} : \frac{a \times g}{l \times w'} :: \text{(in cyl. and sq. prisms,)} \frac{D^3}{L \times W'} : \frac{d^3}{l \times w'}.$$

201. In order that the foregoing general formulæ may be applied to practice, so as to find the actual strength of bars or beams, it is necessary to have some standard of strength ascertained by experiment, which may be employed as the unit of comparison. For example, it is found by experiment that a small beam of oak, one foot long and one inch square, is able, when supported at both ends, to sustain a weight of 600 pounds; and that a bar of iron of the same dimensions, would sustain in the same circumstances 2190 pounds. The beam weighs half a pound, and the iron three pounds. With these data applied to the foregoing formulæ, we may perform such problems as the following.

1. *What weight might be sustained at the middle point of a prismatic beam of oak, whose length is 6 feet, and its end 4 inches square?*

Let S= strength of the beam required.

s= strength of a beam whose length is one foot and square end one inch.

W= weight of the larger beam, and w that of the smaller $=\frac{1}{2}$ pound. Let L=6, l=1, D=4, d=1. Weight required = W′. Given weight (600 pounds) $=w'$.

Then, the weight of the beams not being taken into the account,

$$S : s :: \frac{D^3}{L \times W'} : \frac{d^3}{l \times w'} :: \frac{4^3}{6 \times W'} : \frac{1^3}{1 \times 600}.$$

But the strength, at the moment of fracture = 0 in both cases, i. e. $S=s$; $\therefore \frac{4^3}{6 \times W'} = \frac{1^3}{1 \times 600}$; whence W′=6400 pounds.

If the weight of the beams be taken into the account, then (Art. 200.)

$$S : s :: \frac{D^3}{L \times \frac{1}{2}W + W'} : \frac{d^3}{l \times \frac{1}{2}w + w'} :: \frac{4^3}{6.24^* + W'} : \frac{1^3}{1.\frac{1}{4} + 600}.$$

Hence, $\frac{64}{6.24 + W'} = \frac{1}{600\frac{1}{4}}$; and $W' = 6378\frac{2}{3}$ pounds. Ans.

2. *What must be the depth of a beam of an oblong prismatic form, whose breadth is 2 inches and length 8 feet, to support a weight of 6400 pounds, its own weight not taken into consideration?*

* For $W : w :: L \times D^2 : l \times d^2$, $\therefore W : \frac{1}{2} :: 6 \times 16 : 1 :: 96 : 1$, $\therefore W = 48$.

Here, (Art. 199.) $S : s :: \frac{B \times D^2}{L \times W} : \frac{1^2}{l \times w} :: \frac{2 \times D^2}{8 \times 6400} : \frac{1}{600}$, $\therefore$ $2D^2 = 85.333$, or $D = 6.53$ inches. Ans.

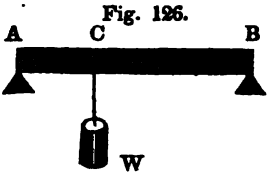
3. *What weight might be supported at the middle point of a bar of iron, 10 feet long, and the side of whose square end is 3 inches, its own weight not being taken into consideration?*

Ans. 5913 pounds.

202. *The stress (or tendency to fracture) on any part of a horizontal beam supported at both ends, is proportioned to the product of its two distances from the supported ends.*

The sum of the pressures on A and B must obviously be equal to the whole weight. But, (Art. 102.)

Fig. 126.



Pressure at $A = \frac{W \times BC}{AB}$, and pressure at $B = \frac{W \times AC}{AB}$.

But the *reaction* of either point of support is equal to the pressure on that point; and this force acts at C with a *leverage* AC on one side, and BC on the other, so that the stress at $C = \frac{W \times BC}{AB} \times AC$ or $\frac{W \times AC}{AB} \times BC$, either of which expressions = stress at C, and $\propto AC \times BC$.

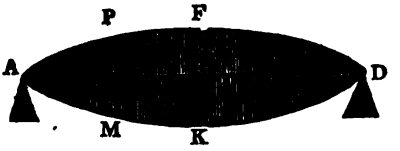
And since this rectangle is greatest when AC = CB, and diminishes as these lines become more and more unequal in length, so, the tendency of a horizontal bar to break is greatest in the middle, and decreases towards the points of support.

Hence a beam, in order to be equally strong throughout, must be thickest in the middle, being thinned off towards the ends; and if the sides of such a beam are parallel planes, the figure of the beam must be *elliptical*.

For let the curve APDM, whose axis is AD, represent this longitudinal section, and let a = the thickness or breadth of the beam;

then a lateral section of the beam at any point C, will be an oblong, whose breadth is a, depth PM, and the depth of its center of gravity $\frac{1}{2}$PM. Hence, the tendency of the beam to resist fracture at any point C, is as $a \times PM^2$; but the *stress* at C is as $AC \times CD$; therefore the strength at $C \propto \frac{a \times PM^2}{AC \times CD} \propto \frac{PM^2}{AC \times CD}$; hence if $PM^2 \propto$

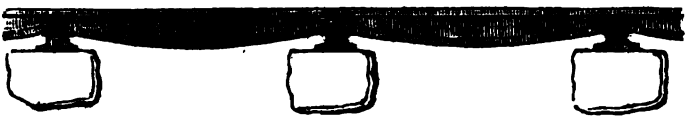
Fig. 127.



$AC \times CD$, the strength will be the same at every point; but in this case the curve APDM is an ellipse, whose transverse is AD, and conjugate FK.

203. The timbers which compose the horizontal part of the frame of a house, being usually rectangular parallelopipeds of uniform dimensions throughout, it is manifest that a considerable portion of the material is wasted; but, in such cases, the attempt to save the material would be attended with paramount disadvantages. When however the material is expensive, or where lightness is important, as in many kinds of machinery, the foregoing principles may be applied with great advantage. A useful application of it is seen in the shape given to the iron bars of railways, as represented in the following figure.

Fig. 128.



204. EXAMPLES.

1. *What must be the length of a beam* 4 *inches square, to support* 6400 *pounds at its middle point?*

Let S = strength of the required beam,

s = strength of a beam 1 foot long, and its end 1 inch square.

W = weight of the larger beam,

w = weight of the smaller beam or $\frac{1}{4}$ lb.

By the question,

L = required length, $l=1$, D = 4, $d=1$.

W′ = 6400 lbs.

w' = weight sustained by the unit = 600 lbs.

(1.) If the weight of the beams be not taken into consideration, by Art. 200,

$$S : s :: \frac{D^3}{L \times W'} : \frac{d^3}{l \times w'} :: \frac{4^3}{L \times 6400} : \frac{1^3}{1 \times 600}.$$

But, at the moment of fracture, the strength in both cases becomes equal to nothing; and then, S being equal to s, $\frac{64}{L \times 6400} = \frac{1}{1 \times 600}$; whence L = 6 feet.

(2.) If the weight of the beams be taken into account, then, (Art. 200.) $S : s :: \frac{D^3}{L(\frac{1}{4}W + W')} : \frac{d^3}{l(\frac{1}{4}w + w')} :: \frac{4^3}{L(4^* + 6400)} : \frac{1^3}{1(\frac{1}{4} + 600)}$,

$\therefore \frac{64}{L(4L + 6400)} = \frac{1}{\frac{1}{4} + 600}$. Whence, L = 5.98 feet.

2. *What must be the breadth of a beam of an oblong prismatic form, whose depth is* 8 *inches and length* 6 *feet, to support a weight of* 6400 *lbs., its own weight not being taken into the account?*

Let B = required breadth.

b = breadth of the beam 1 foot long, and its end one inch square.

Then, (Art. 200.)

$S : s :: \frac{B \times D^3}{L \times W'} : \frac{b^3}{l \times w'} :: \frac{B \times 64}{6 \times 6400} : \frac{1^3}{1 \times 600}$. Hence, at the moment of fracture, when $S = s$, $\frac{B \times 64}{6 \times 6400} = \frac{1}{600}$, whence B = 1 inch. In the two preceding examples, the beams are of the same length, and have equal strengths, each supporting a weight of 6400 lbs. But in beams of equal length, the solid contents are as the areas of the sections. In example 1st, the section $= 4^2 = 16$; and in example 2d, the section $= 1 \times 8 = 8$. Hence,

* The weights being as the solid contents,

$W : w :: L \times D^2 : l \times d^2 :: L \times 16 : 1 \times 1$, $\therefore W : \frac{1}{4} :: L \times 16 : 1$, $\therefore \frac{1}{4}W = L \times 4$.

The first beam : second beam : : 16 : 8. Therefore, the oblong beam placed edgeways, is as strong as the square one, although it contains only one half as much material.

3. *What weight may be supported at the middle of a bar of iron, 10 feet long, and the side of whose square is 3 inches, its own weight being taken into the account?*

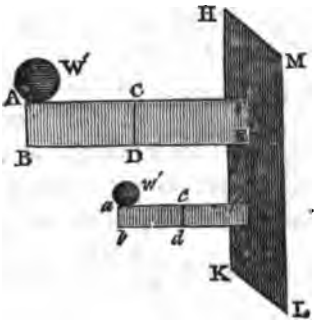
$$S : s :: \frac{D^3}{L(\frac{1}{2}W+W')} : \frac{d^3}{l(\frac{1}{2}w+w')} :: \frac{3^3}{10(135^*+W')} : \frac{1^3}{1(1\frac{1}{2}+2190)}.$$

Hence, $W' = 5782.05$ lbs.

205. The foregoing investigations and examples relate to beams supported at *both* ends: we proceed to the case where the beam is supported at only *one* end.

In similar cylindrical and prismatic beams, supported at one end, *the strength varies inversely either as the diameter or as the length.*

Fig. 129.



Let ABEF, *abef*, (Fig. 129.) represent the longitudinal sections of two prismatic beams fixed horizontally into the wall HKLM; then the tendency of these beams to *resist* fracture at the ends EF, *ef*, where they are inserted into the wall, will be measured in the same manner as in the preceding cases, that is, *by the area of the lateral section into the depth of its center of gravity,* except that in this case, the fracture will begin at the upper points F, *f*, and end at the lower points E, *e*. The tendency to *produce* fracture will

* The bar of iron weighs 270 lbs.

be the weight of the beams acting at the distance of their centers of gravity from the ends EF, *ef*. Hence, (Art. 196.)

$S : s :: \frac{A \times G}{\frac{1}{2}L \times W} : \frac{a \times g}{\frac{1}{2}l \times w}$, or if any weights, W′, *w*′, are placed at the other ends of the beams, then (since the effects of these weights to produce fracture will be measured by $W' \times L$ and $w' \times l$) we have $S : s :: \frac{A \times G}{L\frac{1}{2}W + W'} : \frac{a \times g}{l \times \frac{1}{2}w + w'}$; and if the weights W, *w*, of the beams are very small when compared with the weights W′, *w*′, then $S : s :: \frac{A \times G}{L \times W'} : \frac{a \times g}{l \times w'}$.

Hence, in similar beams, as in Art. 199, $S : s :: \frac{1}{D} : \frac{1}{d}$ or $\frac{1}{L} : \frac{1}{l}$.

Let W, *w*, represent the weights of the parts ABCD, *abcd*, of the beams, then the tendency of those parts to produce fracture at C, *c*, will be measured by $\frac{1}{2}AC \times W$, and $\frac{1}{2}ac \times w$; therefore, if S, *s* represent the strength of the beams at the points C, *c*, then $S : s :: \frac{A \times G}{AC \times \frac{1}{2}W + W'} : \frac{a \times g}{ac \times \frac{1}{2}w + w'}$; or if W, *w*, be very small with respect to W′, *w*′, then $S : s :: \frac{A \times G}{AC \times W'} : \frac{a \times g}{ac \times w'}$.

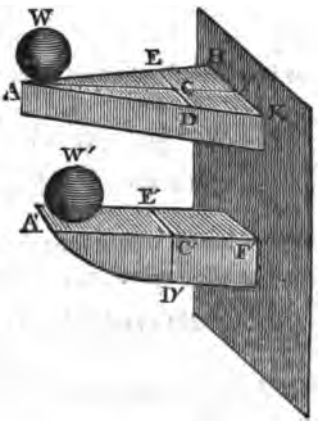
Hence if a given weight W′ be supported at the end of a given beam whose weight is so small as not to be taken into consideration, the strength of that beam to support the weight W′ at any point C, between A and F, will vary as $\frac{A \times G}{AC \times W'}$; or, since W′ is constant, as $\frac{A \times G}{AC}$.

A beam supported at one end in the form of an ISOSCELES WEDGE, *or of a* PARABOLA, *is equally strong throughout; or, when a weight is hung at the end, the beam is as liable to break in one place as in another.*

Let the beam be in the form of an *isosceles wedge*, (Fig. 180.) whose flat sides are parallel to the horizon, and whose given depth $= d$, then $A = ED \times d$, and $G = \frac{1}{2}d$ $\therefore A \times G = \frac{1}{2}ED \times d^2$, which varies as ED or EC, which varies as AC. Hence the strength is as $\frac{AC}{AC}$, that is, it is constant.

If the sides of the beam be parallel planes, and its longitudinal section a semiparabola (see Fig. 130.) as $A'C'F'D'$; then let d equal the given breadth, and we have $A=d\times C'D'$, $G=\frac{1}{2}C'D'$,

Fig. 130.



$\therefore A\times G=\frac{1}{2}d\times C'D'^2$, which varies as $C'D'^2$. Hence $S\propto\frac{C'D'^2}{A'C'}$. But since $A'D'$ is a parabola, $A'C'\propto C'D'^2$, $\therefore S\propto\frac{C'D'^2}{C'D'^2}$.

206. Suppose the beams to be cylinders or square prisms, (resting either on one end or on both ends,) whose diameters or sides are D, d, then $S:s::\frac{D^3}{L\times\frac{1}{2}W+W'}:\frac{d^3}{l\times\frac{1}{2}w+w'}$. Let $w'=0$, then $S:s::\frac{D^3}{L\times\frac{1}{2}W+W'}:\frac{d^3}{\frac{1}{2}l\times w}$, which expresses the relative strength of two cylindrical beams whose lengths are L, l, diameters D, d, weights W, w, the former of which supports the given weight W' at the end of it, and the latter supports only its own weight. Let $d=D$; then $S:s::\frac{1}{L\times\frac{1}{2}W+W'}:\frac{1}{\frac{1}{2}l\times w}::\frac{1}{L\times\frac{1}{2}W+W'}:\frac{2L}{W\times l^2}$: for, since $w:W::l:L$, $\therefore w=\frac{W\times l}{L}$. In this case, therefore, the ratio of $S:s$ expresses the relative strength of two cylindrical beams of the same diameter, one of which supports the given weight W' at its end, and the other supports only its own weight; and if the beam whose length is L breaks when W' is placed at the end of it, the beam

whose length is l will break by its own weight. Hence, let $S=s$, then $\frac{1}{L.\frac{1}{2}W+W'}=\frac{2L}{W\times l^2}$ $\therefore l^2=\frac{L^2(W+2W')}{W}$ $\therefore l=L\sqrt{\frac{W+2W'}{W}}=$ length of the beam of the same diameter that would break by its own weight.

207. Let the beams be *similar* cylinders, then $D^2 : d^2 :: L^2 : l^2$, $\therefore$ (Art. 206.) $S : s :: \frac{L^2}{\frac{1}{2}W+W'} : \frac{l^2}{\frac{1}{2}w} :: \frac{L^2}{\frac{1}{2}W+W'} : \frac{2L^2*}{W\times l}$ And when $S=s$, $\frac{L^2}{\frac{1}{2}W+W'}=\frac{2L^2}{W\times l}$ $\therefore \frac{1}{2}W+W'=\frac{W\times l}{2L}$, or $l=\frac{L(W+2W')}{W}$. If, therefore, a cylindrical beam whose length is L breaks with the given weight W' placed at the end of it, a similar cylindrical beam whose length is $\frac{L(W+2W')}{W}$ will break with its own weight.

208. If a horizontal beam be supported at *both ends*, the stress produced by its own weight, W, is measured by $\frac{1}{2}L\times W$, (Art. 196.)

If the beam be supported at *one end* only, the stress is measured by the whole weight applied at the center of gravity, and consequently the stress $=\frac{1}{2}L\times W$.

Therefore a beam supported at both ends has four times the strength of the same beam, supported only at one end. And if a certain beam resting on one end breaks by its own weight, a beam of the same dimensions twice as long will break by its own weight when resting on two supports, the former having just four times the strength it would have if twice as long.

If, however, instead of the weight of the beam itself, this be left out of the account, and a weight W' be appended, then the stress on the beam when supported at *one end* will be measured by $L\times W'$; while, in the case of the beam supported at *both ends*, (since the weight, being at the centre is also at the centre of gravity of the beam) the stress is measured, as before, by $\frac{1}{4}L\times W'$. (Art. 200.) Therefore, a weight appended at the end of a beam supported only at one end produces four times the stress, as the same weight applied at the centre of the beam when supported at both ends.

* The weights of similar cylinders of the same density, are as the cubes of their diameters or lengths; therefore, $w : W :: l^3 : L^3$ $\therefore w=\frac{Wl^3}{L^3}$.

209. Examples.

1. *What must be the length of a beam of oak one inch square, supported at both ends, which is just capable of bearing its own weight?*

By Art. 201, a beam of oak 1 foot long and 1 inch square, weighing $\frac{1}{2}$ pound, just supports 600 pounds. And by Art. 206, the expression $l = \mathrm{L}\left(\frac{\mathrm{W}+2\mathrm{W}'}{\mathrm{W}}\right)^{\frac{1}{2}}$ denotes that when a beam whose length is L breaks when W' is placed at the end of it, l is the length of a beam that will break with its own weight, consequently, since here $\mathrm{L} = 1$, $\mathrm{W} = \frac{1}{2}$, and $\mathrm{W}' = 600$, $l = \left(\frac{\frac{1}{2}+1200}{\frac{1}{2}}\right)^{\frac{1}{2}} = (2401)^{\frac{1}{2}} = 49$ feet.

2. *What must be the length of a bar of iron 1 inch square, supported at one end, which would break by its own weight?*

Here L=1 foot, W=3 pounds, and since (Art. 208.) a beam supported at one end will break with $\frac{1}{4}$ as great a weight as when supported at both ends, $\mathrm{W}' = 547\frac{1}{2}$ pounds, $\therefore$

$$l = 1\left(\frac{3+2\times 547\frac{1}{2}}{3}\right)^{\frac{1}{2}} = 19.13.$$

3. *Since a bar of iron 1 inch square, and 1 foot long, will support a weight of 2190 pounds, what must be the dimensions and weight of a similar bar, which will break with its own weight when supported at both ends?*

The required beam being *similar* to the given beam, therefore, by Art. 207, its length equals 1461 feet. And $\mathrm{L}(1) : l(1461) :: \mathrm{D}(1\text{ inch}) : d = 121\frac{3}{4}$ feet=the side of its square. Again, since the weights are as the cubes of the homologous sides, $w : \mathrm{W}(3) :: l^3(1461)^3 : 1$, $\therefore$ the weight = 9355605543 pounds.

4. *Two beams are of equal length and weight, the first being a square prism whose section is 4 inches square, the second an oblong 8 by 2 inches: How much stronger is the second beam than the first, and how much stronger when laid on the narrow than on the broad side?*

Ans. The second beam is TWICE as strong as the first, and FOUR times as strong when laid on the narrow, as on the broad side.

5. *A beam of oak, supported at both ends, is one inch square and 40 feet long: At what point of the beam will a weight of 100 pounds, be just sufficient to break it?*

Let x=distance of the weight W′ from one end; $W' \times 400$ represents the stress at the middle, and (Art. 202.) $W' \times \frac{x(40-x)}{400}$ the stress at the required point.

Let l be the length of a beam one inch square that would break with its own weight=49 feet (Art. 209. Ex. 1.)

By Art. 206, $\frac{1}{2}l \times w = L\left(\frac{1}{2}W + W' \times \frac{x(40-x)}{400}\right)$. Hence x=.51 feet, or about 6 inches.

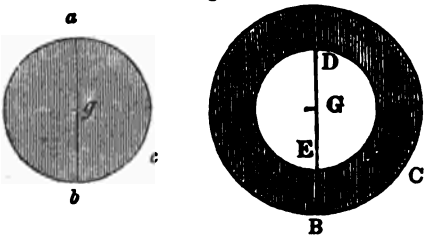
210. On the foregoing principles Dr. Gregory makes the following remarks, most of which were originally suggested by Galileo, to whom we are indebted for the earliest investigation of these propositions. From the preceding deductions (says Gregory) it follows, that greater beams and bars must be in greater danger of breaking than less similar ones; and that, though a less beam may be firm and secure, yet a greater similar one may be made so long as necessarily to break by its own weight. Hence Galileo justly concludes, that what appears very firm, and succeeds well, in models, may be very weak and unstable, or even fall to pieces by its weight, when it comes to be executed in large dimensions, according to the model. From the same principles he argues that there are necessarily limits in the works of nature and art, which they cannot surpass in magnitude; that immensely great ships, palaces, temples, &c., cannot be erected, since their yards, beams, bolts, and other parts of their frame, would fall asunder by their own weight. Were trees of a very enormous magnitude, their branches would, in like manner, fall off. Large animals have not strength in proportion to their size; and if there were any land animals much larger than those we know, they could hardly move, and would be perpetually subjected to the most dangerous accidents. As to marine animals, indeed, the case is different, as the specific gravity of the water sustains those animals in a great measure; and in fact these are known to be sometimes vastly larger than the greatest land animals.* It is (says Galileo) impossible for Nature to give bones to men, horses, or other animals, so formed as to subsist, and proportionally to perform their offices, when such animals should be enlarged to immense heights,

* Whales in the Northern Regions, are sometimes found sixty feet long, and weighing seventy tons.

unless she uses matter much firmer and more resisting than she commonly does; or should make bones of a thickness out of all proportion; whence the appearance and figure of the animal must be monstrous. Hence, we naturally join the idea of greater strength and force with the grosser proportions, and that of agility with the more delicate ones. The same admirable philosopher likewise remarks, in connection with this subject, that a greater column is in much more danger of being broken by a fall than a similar small one; that a man is in greater danger from accidents than a child; that an insect can sustain a weight many times greater than itself, whereas, a much larger animal, as a horse, could scarcely carry another horse of his own size.*

211. *The lateral strengths of two cylinders, of the same matter, and of equal weight and length, one of which is hollow and the other solid, are to each other as the diameters of their sections.*

Fig. 131.



Let ABC, *abc*, represent a section of two cylinders, of equal length, and containing equal quantities of matter, of which ABC is hollow and *abc* is solid. Then the area of the ring whose breadth is AD, is equal to that of the circle *abc*. But the strengths of these areas are as the areas multiplied by the distances of their centers of gravity from the points of pressure, A, *a* (Art. 194.); or, since the areas are equal, the strengths are as AG : *ag*, that is, as the diameters of their sections.

The strongest form, therefore, in which a given quantity of matter can be disposed, is that of a hollow cylinder; and, leaving out of view the diminished rigidity of their structures or fabrics, there would seem to be no limits to the strength which might be given to such a cylinder by increasing its diameter. But the proposition is

* Gregory's Mechanics, I. 110.

true only when the sections are perfectly circular; and this condition, connected with the want of rigidity when the annulus becomes very thin, occasions limits to the actual operation of the principle.

212. From this proposition Galileo justly concludes, that Nature in a thousand operations greatly augments the strength of substances without increasing their weight; as is manifested in the bones of animals, and the feathers of birds, as well as in most tubes or hollow trunks, which, though light, greatly resist any effort to bend them. Thus, (says he,) if a wheat straw, which supports an ear heavier than the whole stalk, were made of the same quantity of matter, but solid, it would bend or break with far greater ease than it now does. And with the same reason, art has observed, and experience confirmed the fact, that a hollow cane, or tube of wood or metal, is much stronger or firmer, than if, while it continues of the same weight and length, it were solid; as it would then, of consequence, be not so thick. For the same reason, lances, when they are required to be both light and strong, are made hollow.*

213. The area ABC : abc :: AB^2 : ab^2.

And DEF : abc :: DE^2 : ab^2

$\therefore$ ABC − DEF : abc : $AB^2 - DE^2$: ab^2.

Or the area of the ring is to the area of the solid section, as $AB^2 - DE^2$: ab^2. If the area of the ring is equal to the solid section, then $AB^2 - DE^2 = ab^2$ and $ab = \sqrt{(AB^2 - DE^2)}$.

What weight could be sustained at the middle point of a cylindrical iron tube 8 *feet long, whose diameter is* $1\frac{1}{2}$ *inches, and thickness* $\frac{1}{4}$ *of an inch; supposing the tube to be supported at both ends?*

The diameter of a solid cylinder of the same length and weight = 1.11 inches, and

$S : s\dagger :: \frac{A \times G}{L \times W'} : \frac{a \times g}{l \times w'} :: \frac{.532}{8 \times W'} : \frac{.3927}{2190}$. Therefore $W' = 370.8$ pounds.

But (Art. 211.) strength of the tube : strength of the cylinder :: diam. tube : diam. cyl. :: $\frac{3}{2}$: 1.11, $\therefore$ The tube would sustain at its middle point a weight of $\frac{3 \times 370.8}{2.22} = 501.08$ pounds.

* Gregory, I. 112.

† s = strength of a bar of iron 1 foot long and 1 inch square.

PART II.—PRACTICAL APPLICATIONS OF THE PRINCIPLES OF MECHANICS.

CHAPTER I.

GENERAL OBSERVATIONS ON MOTION.

214. MOTION and rest are accidental states of bodies, nor is a body naturally prone to one state more than to the other. If it is found at rest, it is because it is kept in equilibrium by opposite and equal forces; and if it is found in motion, it is because it has been put in motion by some force extrinsic to itself. The resistances to motion which exist near the surface of the earth, particularly gravity, create a seeming tendency to a state of rest; but in reality, rest is no more the natural state of bodies than motion is.

215. Motion is distinguished into absolute and relative. *Absolute motion*, is a change of place in space with respect to any fixed point: *Relative motion*, is a change of place in bodies with respect to each other. A body may be at the same time in a state of absolute motion, and of relative rest. Thus, all the different articles contained in a ship under sail, have a motion in common with the ship, but may be at rest with respect to each other. When a man walks towards the stern of a ship at the same rate as that of the ship, he is in motion with respect to the ship, but at rest with respect to the earth. When a balloon, carried along by the wind, attains the same velocity as the wind, it is relatively at rest, and appears to the aeronaut to be in a perfect calm, although it may be actually moving one hundred miles an hour. Since the earth in its annual revolution around the sun, is moving eastward at the rate of nineteen miles, or 100,000 feet per second, were a cannon ball, at a certain time of day, fired eastward at the rate of 2000 feet per second, the only effect would be to add 2000 feet to the velocity which the ball had before in common with the earth; and were it fired westward, the effect would be

merely to stop 2000 out of 100,000 parts of its previous motion, while the cannon would proceed onward leaving it behind.* Did not the atmosphere partake of the diurnal motion of the earth, but were it to remain at rest with respect to this motion, the progress of any place to the east, would cause a relative motion of the air, or a wind, westward, which would blow with a violence far surpassing that of the most terrible hurricanes.†

Indeed we cannot be sure that we have ever seen a body absolutely at rest. In our stillest moments, we are revolving with the earth on its axis; we are accompanying the earth in its annual revolution from west to east around the sun; and are perhaps attending the solar system around a common center of motion.‡

216. *Apparent motion*, as distinguished from relative,' is that in which the moving body is quiescent, and the motion is owing to a real motion in the spectator. Thus, the backward motion of the trees to one riding rapidly, the receding of the shore to one who is sailing from it with a fair wind, and the diurnal motions of the heavenly bodies from east to west, in consequence of the revolution of the spectator in the opposite direction; these are severally examples of apparent motion. It is often a very difficult problem to deduce the real from the apparent motion. While a planet, as Venus, is revolving about the sun in an orbit nearly circular, its motions as seen from the earth are extremely irregular; and to make all these irregularities consistent with the real motion, has been a perplexing problem in astronomy. We can sometimes decide that a given motion is real, because we observe a *cause* in operation, which is competent to produce it. The impulse of the wind, or the direction of the current, will satisfactorily account for a ship's receding from a given object, while no cause appears why the object should recede from the ship; the revolution of the earth on its axis is a cause competent to explain the apparent revolution of the heavens, while we can find no cause for their actual revolution. The *effects* also of a given motion, enable us to decide whether it is real or apparent.

* Robison's Mechanical Phil. I. 31.

† Winds are in fact frequently produced by this cause, viz. by their having a relative velocity different from that of the part of the earth over which they blow.

‡ Young's Natural Phil. I. 19.

Thus a constant tendency to move in a straight line is characteristic of real motion.*

217. The LAWS OF MOTION have been already recited in Chapter I., and concise illustrations of them were added in that place. It was necessary to proceed thus far at the beginning of this work, since these constitute the fundamental principles of mechanics. By their great comprehensiveness, they furnish the most convenient classification of the various phenomena of motion, and it will therefore be useful to resume the consideration of them. They are very remarkable examples of a happy generalization; but their very comprehensiveness renders them difficult to be understood by the young learner; nor can they be thoroughly mastered in all their relations, until after considerable proficiency is made in the science of mechanics. These laws indeed are the chief foundation of Newton's Principia.†

218. FIRST LAW.—*A body continues always in a state of rest, or of uniform rectilinear motion, until by some external force, it is made to change its state.*—This law contains the doctrine of INERTIA, expressed in four particulars. First, that unless put in motion by some external force, a body always remains at rest; secondly, that when once in motion it continues always in motion, unless stopped by some force; thirdly, that this motion, arising from INERTIA, is uniform; and, fourthly, that this motion is in right lines. The proofs by which this and the other laws of motion are established, have been already stated. (Art. 22.) It is our present object to make the application of these laws to various phenomena of nature and art.

219. And first, with respect to bodies *at rest*. The operation of this principle is seen, when a horse starts suddenly forward, and his rider is thrown backward. "When we desire a person, with suspected disease of the brain, to shake his head, and tell whether he feels pain, we are doing nearly the same as if we touched the naked brain with the finger to find the tender part; for the Inertia of the brain, when the skull is moved, causes a momentary pressure between it and

* Wood's Mechanics, p. 22. † Young's N. Phil. I. 26.

the skull, almost equivalent for our purpose to such a touch."* In consequence of the Inertia of matter, before a body can be brought to the required velocity, this velocity must be impressed upon every particle of matter it contains. Hence, the more numerous its particles, the greater its Inertia, which is therefore proportioned to the quantity of matter. But the weight also is proportioned to the quantity of matter, and therefore the Inertia is proportioned to the weight. Yet it must be carefully distinguished from weight, having in fact nothing in common with it, except that both are proportioned to the quantity of matter, and of course to each other. But were we to strike with a hammer upon the top of a body falling towards the earth, the resistance from Inertia would be the same as if the body were struck with the same force on the side; or in whatever direction the blow were applied, a similar resistance would be felt. This seems little else than what we commonly understand by the reaction of a body; but we conceive this reaction itself to depend upon an inherent property in matter, to which we give the name of Inertia. Inertia is the *cause* and reaction the *effect*. A vast weight may be moved on a horizontal railway by a comparatively small force, provided it can be got into motion with the required velocity. In transporting large quantities (eighty tons, for instance,) of coal, the weight is distributed into a number of different cars, connected together by a loose chain, in order that the Inertia of the several parts may be overcome successively.†

220. In consequence of the Inertia of matter, the motion applied to a body does not instantly pervade the mass. In order to this, motion must be applied gradually, especially if the body is large; for if it is applied suddenly, it is frequently all expended on a part only of the mass, the cohesion is overcome, and the body is broken. This explanation accounts for several familiar facts. When a team starts suddenly with a heavy load, the effort is either wholly ineffectual, or some part of the harness or tackling gives way. If we draw a heavy weight by a slender string, a slow and steady pull will move the weight, when a sudden twitch would break the string without starting

* Arnott's El. Phys. p. 50.

† See account of the Hatton Railway in "Strickland's Reports."

the mass. The same principle applies to bodies already in motion. Thus, when a horse in a carriage starts suddenly forward, he may break loose as well when the carriage was previously in motion, as when it was at rest. The Inertia of a body is in fact the same whether the body is in motion or at rest, opposing the same resistance to its moving with increased velocity, as to its beginning to move from a state of rest. Several singular phenomena result from the same cause, showing that time is necessary in order that motion communicated by impulse, may pervade an entire mass. A pistol ball, fired through a pane of glass, frequently makes a smooth well defined hole, and does not fracture the other parts of the glass. Here, the momentum of the ball is communicated to the particles of glass immediately before it. Had the impulse been gradual, the same motion would have diffused itself over the whole pane, and every part would have felt the shock. A ball fired through a board delicately suspended, causes no vibrations in the board. A cannon ball, having very great velocity, passes through a ship's side, and leaves but little mark, while one with less speed, splinters and breaks the wood to a considerable distance around. A near shot thus often injures a ship less than one from a greater distance.*

A soft substance, as clay or tallow, may be fired through a plank, the body, by its great momentum, forcing its way through the plank before the motion has had time to diffuse itself through the contiguous parts. The whole momentum being concentrated upon the part immediately before the body, the cohesion of that part is destroyed.

221. *Secondly*, let us consider the effects of Inertia as it respects bodies *in motion*. All bodies in contact with each other acquire a common motion; as, for example, a horse and his rider, a ferry boat and its passengers, a ship and every thing within it, the earth and all things on its surface. Whenever either of these bodies stops suddenly, the movable bodies connected with it are thrown forward.—Were the revolution of the earth on its axis to be suddenly arrested, the most dreadful consequences would ensue; every thing movable on its surface, as water, rocks, cities, and animals, not receiving instantaneously this backward impulse, would fly off eastward in pro-

* Arnott's El. Phys. p. 104.

miscuous ruin. Were the diurnal motion of the earth, however, very gradually diminished, until it finally ceased, so that time should be afforded to communicate the loss by slow degrees to the bodies on its surface, no such effects would take place. If a passenger leaps from a carriage in rapid motion, he will fall in the direction in which the carriage is moving at the moment his feet meet the ground; because his body, on quitting the vehicle, retains, by its Inertia, the motion which it had in common with it. When he reaches the ground, this motion is destroyed by the resistance of the ground to the feet, but is retained in the upper and heavier part of the body, so that the same effect is produced as though the feet had been tripped.* *Coursing* owes all its interest to the instinctive consciousness of the nature of Inertia, which seems to govern the measures of the hare. The greyhound is a comparatively heavy body moving at the same or greater speed in pursuit. The hare *doubles*, that is, suddenly changes the direction of her course, and turns back at an oblique angle with the direction in which she had been running. The greyhound, unable to resist the tendency of its body to persevere in the rapid motion it had acquired, is urged forward many yards before it is able to check its speed and return to the pursuit. Meanwhile, the hare is gaining ground in the other direction, so that the animals are at a very considerable distance asunder when the pursuit is recommenced. In this way a hare, though much less fleet than a greyhound, will often escape it.†

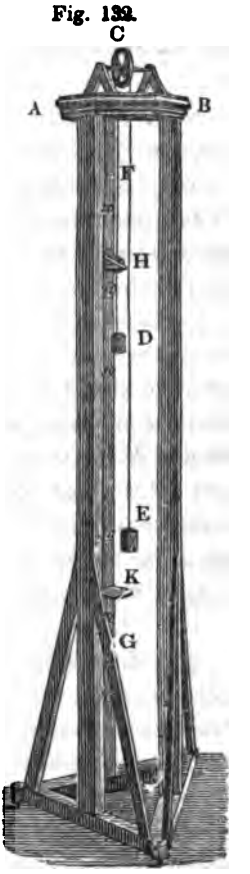
222. Although, on account of the numerous impediments to motion which exist on the surface of the earth, bodies are unable to maintain for any considerable time, the motion they have acquired, yet we see the first law of motion, so far as it respects the tendency of bodies to persevere in motion, fully confirmed in the continued and unaltered revolution of the heavenly bodies. These are impelled by no renewed forces, but revolve from age to age in an undeviating course, simply because they meet with no impediments.‡

* Lardner's Mechanics, p. 34. † Ib.

‡ Refined observations have recently detected a slight resistance, and brought to light the existence of an extremely rare medium diffused through the celestial spaces.

223. *Thirdly*, bodies, in consequence of their Inertia, have a tendency to move over equal spaces in equal times, that is, to move uniformly. In a ball rolled on ice, in a pendulum continuing to vibrate after the moving force is withdrawn, and in numerous cases similar to these, we observe in nature and art this tendency to uniform motion; but in all these cases, the motion is not absolutely uniform, but more or less retarded by the resistances encountered. A much nearer approximation to the truth is obtained by means of a piece of apparatus called *Atwood's Machine*. Its construction, omitting some parts not essential to the principle, is as follows. The triangular base and upright pillars (which are usually of mahogany) constitute the *frame*, which is surmounted by a horizontal table or plate of wood AB, Fig. 132, perforated with several holes. C is a vertical wheel, which, by a contrivance called *friction wheels*, (not represented in the figure,) is made to revolve with the least possible resistance from friction. D and E are two weights exactly equal, and connected by a slender string passing over the wheel C. FG is a perpendicular scale graduated into inches from top to bottom, extending from 0 to 60 or 70, according to the height of the machine. H is a movable ring which slides up and down on the scale, and K is a brass plate sliding in the same manner.* A great variety of the principles of motion may be established by means of this apparatus, but we are at present concerned only with the method of showing, that a body when once put in motion continues, by its inertia, to move uniformly, after the moving force is withdrawn. It is obvious that the weights D and E balance each other, and consequently, that the power of gravity is entirely removed from D, so that it is at liberty to obey the full and exclusive influence of any force that may be applied to it. If therefore an impulse be given,

Fig. 132.



* There are frequently connected with the machine a pendulum, and such parts of a clock as are necessary for beating seconds.

by the finger, for example, to D, when at the top of the scale, it ought, in conformity to the law under consideration, to move uniformly along down the scale, passing over the same number of inches in each successive second. Such appears to be the fact. But in order to give still greater precision to the experiment, a small brass bar is laid on D, which communicates motion to it, accelerating its progress until it comes to the brass ring H, where the bar lodges, and the weight D, after it leaves the ring, passes accurately over the same number of inches on the scale in each successive second.

224. *Fourthly*, moving bodies have a constant tendency to proceed in right lines. In nature there occur indeed but few examples of rectilinear motion, but almost every moving body describes a curve. Thus, the heavenly bodies move in ellipses; projectiles describe parabolas; or if their direction is so altered by a resisting medium, as the atmosphere, that their path is no longer a parabola, it is still changed to some other curve; and a ship sailing across the ocean, describes a curvilinear path on the surface of the earth. The waving of trees and plants, the courses of rivers, the spouting of fluids, the motions of winds and waves, are likewise more or less curvilinear. Bodies falling towards the earth by gravity, present almost the only examples we observe in nature of a motion purely rectilinear. But notwithstanding the deviation from a right line, observable in actual motions, yet we find there is always some extraneous cause in operation, which accounts for such deviations.

225. In consequence of this tendency of moving bodies to proceed in right lines, when a body revolves in a curve, around some center of motion, it constantly tends to fly off in a straight line, which is a tangent to its orbit. This is called the *centrifugal force*. A stone from a sling, water escaping from the periphery of a revolving wheel, and water receding from the center of a tumbler or pail when the vessel is whirled, are familiar instances of the tendency of bodies when revolving in circles to fly off in straight lines. The action of the centrifugal force may be studied more advantageously by means of an apparatus called the Whirling Tables. These consist of two small circular tables, to which is communicated a horizontal revolution around their centers. Bodies laid on the tables in different ways are

made to participate in their rotary motions, and thus the laws of the centrifugal force may be observed. By means of this apparatus the following propositions are established.

226. (1.) The centrifugal force of bodies revolving in a given circle, is proportional to their *densities or specific gravities.* If quicksilver, water, and cork, be whirled together in a tube or a vessel, these bodies will arrange themselves in the inverse order of their specific gravities, so that the cork will be at the least, and the quicksilver at the greatest distance from the center of motion.

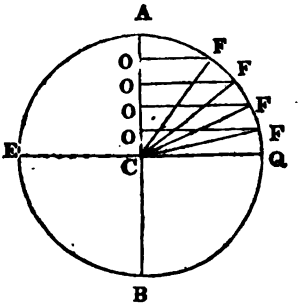
227. (2.) When bodies revolve in the same circle with different velocities, the centrifugal forces are as the *squares of the velocities.* By doubling the velocity of a revolving body, its centrifugal force is quadrupled. Millstones, revolving horizontally, communicate their circular motion to the corn that is introduced between them near the center. The corn, by the centrifugal force which it gradually acquires, recedes from the center and passes out at the circumference. If too great velocity is given to millstones, they sometimes burst with violence. A horse in swift motion, on suddenly turning a corner, throws his rider; and a carriage turning swiftly is overset on the same principle. In feats of horsemanship, when the equestrian rides rapidly round a small ring, he inclines his body inwards in different degrees according to the velocity with which he is moving, and thus counteracts his tendency to fall outwards by the centrifugal force.*

228. (3.) Hence, when spherical bodies revolve on their axes, the equatorial parts, being farthest from their centers of motion, and consequently moving faster, have a proportionally greater centrifugal force. If the revolving body is soft so as to yield, it is elevated in the equatorial and depressed in the polar parts. Thus a mass of clay revolving on a potter's wheel, swells out in the central parts and becomes flattened at the two ends. The earth itself, by its figure, which is an oblate spheroid, indicates the operation of this principle, its equatorial exceeding its polar diameter by 26 miles; and the planet Saturn, which has a far more rapid revolution on its axis, indicates

* Arnott's Elements of Phys. p. 62.

the same modification of its figure in a still higher degree, being strikingly elevated at the equator and depressed at the poles.* Let the circle in Fig. 133, represent a section of the earth, AB being the axis on which it revolves. This rotation causes the matter, which

Fig. 133.



composes the mass of the earth, to revolve in circles round the different points of the axis as centers, at the various distances at which the component parts of the mass are placed. As they all revolve with the same angular velocity, they will be affected by centrifugal forces, which will be greater or less in proportion as their distances from the center are greater or less. Consequently, the parts of the earth which are situated about the equator Q, will be more strongly affected by centrifugal force than those about the poles AB: the effect of the difference has been, that the matter about the equator has actually been driven farther from the center than that about the poles, so that the figure of the earth has swelled out at the sides, and appears proportionally depressed at the top and bottom, resembling an orange in shape.

229. The centrifugal force of the earth's rotation also affects detached bodies on its surface. If such bodies were not held upon the surface by the earth's attraction, they would be immediately flung off by the whirling motion in which they participate. The centrifugal force, however, really diminishes the effects of the earth's attraction on those bodies, or, what is the same, diminishes their weight. If the earth were not revolving on its axis, the weight of bodies in all places equally distant from the center would be the same; but this is not so when the bodies, as they do, move round with the earth.

* Lardner's Elements of Mechanics, p. 105.

They acquire from the centrifugal force a tendency to fly off from the axis, which increases as their distance from that axis, and is therefore greater the nearer they are to the equator, and less as they approach the pole. But there is another reason why the centrifugal force is more efficient in the opposition which it gives to gravity near the equator than near the poles. This force does not act from the center of the earth, but is directed from the earth's axis. It is therefore, not directly opposed to gravity except on the equator itself. On leaving the equator, and proceeding towards the poles, it is less and less opposed to gravity, as will be plain on inspecting Fig. 133, where the lines FC all represent the direction of gravity, and the lines OF represent the direction of the centrifugal force.* If the diurnal motion of the earth round its axis were about seventeen times faster than it is, the centrifugal force would, at the equator, be equal to the power of gravity, and all bodies there would entirely lose their weight. But if the earth revolved still quicker than this, they would all fly off.†

230. The consideration of centrifugal force proves, that if a body be observed to move in a curvilinear path, some efficient cause must exist which prevents it from flying off, and which compels it to revolve round the center. Thus the bodies of the solar system are constantly impelled or drawn towards the sun by a force which we denominate gravity. If this force did not act constantly, they would resume their motion in the right line in which they were originally projected, when they were first launched into space, and continue moving in it forever.

231. SECOND LAW.—*Motion, or change of motion, is proportional to the force impressed, and is produced in the right line in which that force acts.*

First, motion is proportional to the force impressed. This is very satisfactorily shewn by means of Atwood's machine, (Art. 223. Fig. 132.) When the box D is loaded with small bars of different weights, (the bars being left on the ring H, as in Art. 223.) the box

* At the equator, the loss of weight by the centrifugal alone is $\frac{1}{289}$; and by increased distance from the center of the earth, $\frac{1}{578}$; hence the entire loss is $\frac{1}{193}$.

† Boucharlat's Mech. by Courtenay, p. 251.

descends along the scale, in consequence of the motion given it by the bars, with velocities exactly proportional to the weights of the bars respectively.

Secondly, motion is in the *direction* of the force impressed. Notwithstanding the diversity of motions to which every terrestrial body is constantly subject, the effect of any force to produce motion is the same, when the spectator has the same motion as the body, as though the body were absolutely at rest. In other words, all motions are compounded so as not to disturb each other; each remaining, relatively, the same as if there were no others.* Since, for example, by the diurnal motion of the earth, places towards the equator move faster than those towards the poles, if the foregoing principle were not true, the same forces would produce different quantities of motion in different latitudes; and a body struck in a direction north or south, would not move in that direction, but would deviate to the east or west. A pendulum also would vibrate differently accordingly as it moved in a north and south, or in an east and west direction, whereas not the slightest difference of time can now be detected. If we are in a ship, moving equally, any force which we can exert will produce the same motion relative to the vessel whether it be or be not in the direction of the vessel's motion. If we stand on the deck, supposed to be level, and roll a body along it, the same effort will produce the same velocity along the deck whether the motion be from head to stern, or from stern to head, or across the vessel. Also a body dropped from the top of the mast will not be left behind by the motion of the ship, but will fall along the mast as it would if the mast were at rest, and will reach the foot of it in the same time. If a body be thrown perpendicularly upwards, it will rise directly over the hand and fall perpendicularly upon it again; and if it be thrown in any other direction, the path and motion relative to the person who throws it will be the same as if he were at rest.†

232. It may seem, at first view, more questionable whether, as is asserted in Art. 20, *the smallest force is capable of moving the largest body*. Agreeably to this doctrine, a blow with a hammer upon

* Whewell's Mechanics, p. 231.

† Robison's Mech. Phil. by Brewster, I. 42.

the earth ought to move it, and that it would do so may be inferred from the following reasons.

(1.) We can conceive the earth to be divided into parts so small, that the blow would produce upon one of them even a sensible motion. Then it would produce on two of the parts half as much velocity; and upon all the parts together a velocity as much less than upon one, as their number was greater than unity. This velocity might be appreciable in numbers, although too small to be observed by the senses.

(2.) Very heavy weights may be actually put in motion by small forces. Leslie asserts that a ship of any burden in calm weather and smooth water, may be gradually pulled along even by the exertions of a boy.*

(3.) The repetition of very small blows finally produces sensible effects upon large bodies. The wearing away of stone by the dropping of water, the abrasion of marble images by the kisses of pilgrims, and, especially, the demolition of the strongest fortresses by repeated blows of the battering ram, are examples of powerful effects produced by small impulses, each of which must have contributed its share, since the addition of any number of nothings is nothing still.

233. THIRD LAW.—*When bodies act upon each other, action and reaction are equal, and in opposite directions.*

The doctrine of action and reaction has been fully investigated and explained in the former part of this work. (See pp. 15—17, and 81—9.) All we propose to do at present is to add a few familiar illustrations.

234. If I strike one hand upon the other at rest, I perceive no difference in the sensations experienced by each. The resistance to the hand which gives the blow is equal to the impulse given. A boatman presses against the bank with his oar, and receives motion in the opposite direction, which being communicated to the boat, makes it recede from the shore. He strikes the water, the reaction of which, at every impulse, carries the boat forward in the opposite direction. An infirm old man presses the ground with his staff, and

* Leslie, El. Nat. Phil. I. 30.

thus, by lightening the pressure on his lower limbs, makes his arms perform a part of the labor of walking. A bird beats the air with his wings, and by giving a blow whose reaction is more than sufficient to balance the weight of his body, rises with the difference. When the wings are small and slender, as those of the humming bird, and disproportioned to the weight of the body, the defect is compensated by more frequent blows, giving nimble motions, suited to their short but swift excursions, while the long wings of the eagle are equally fitted, by their less rapid but more effectual blows, for his distant journeys through the skies. Hence, propelling and rowing a boat, flying, and swimming, are processes analogous to each other, depending on the principle of reaction.

235. If a man stands in a boat and pulls upon a rope which is fastened to a post on the shore, the force of the man is expended on the post in one direction, and the post, by its reaction, draws the man in the opposite direction, namely, towards the shore. (See p. 19. Ex. 12.) Call the man A, and let another man B take the place of the post. If B pulls with a force just equal to that of A, he will do nothing more than what the post did before, and therefore the two men together will bring the boat ashore no sooner than A would have done alone in the former case. If A pulls with more force than B, he pulls B towards him, and the reaction, or the force which carries the boat ashore, is the same as before, namely, the force of B. If B were to pull with more force than A, he would pull A out of the boat, were not A attached firmly to the boat, in which case the velocity of the boat would be augmented. By attentively considering this and all analogous cases, we shall perceive, that whenever two bodies act against each other, they give and receive equal momenta, and the momenta being in opposite directions, it follows, that bodies do not alter the quantity of motion they have, estimated in a given direction, by their mutual action on each other. This principle is well explained in Emerson's Mechanics, as follows—

*The sum of the motions of any two bodies in any one line of direction, towards the same part, cannot be changed, by any action of the bodies upon each other; whatever force these actions are caused by, or the bodies exert among themselves.** Hence it follows, that

* Emerson's Mechanics, 4to. p. 17.

the sum of the motions of all the bodies in the world, estimated in one and the same line of direction, and always the same way, is eternally and invariably the same. Whatever motion, therefore, one body receives towards another, whether it is drawn towards it by attraction, or by a rope, or by any other method, precisely the same quantity of motion it imparts to the other body in the opposite direction.* If a man in a boat pulls at a rope attached to another boat of equal size, the boats will move towards each other with equal velocities; but a man in a boat pulling a rope attached to a large ship seems only to move the boat, but he really moves the ship a little, although its velocity is as much less than that of the boat, as its weight is greater. A pound of lead and the earth attract each other with equal force, and the two bodies approach each other with equal momenta.†

236. Since momentum is proportioned to the joint product of the velocity and quantity of matter, a great momentum may be obtained either by giving a slow motion to a great mass, or a swift motion to a small body. A striking illustration of this is afforded by problem 9th, page 19th, where on the supposition that a grain of light moving with its usual velocity, were to impinge directly against a mass of ice floating at its ordinary slow rate, the grain of light would be competent to stop about 44½ tons of ice. Islands of ice move with such vast momentum, that they instantly demolish the largest ship of war if it comes in their way.

237. If a body in motion strikes a body at rest, the striking body must sustain as great a shock from the collision as if it had been at rest, and struck by the other body with the same force. For the loss of force which it sustains in one direction, is an effect of the same kind as if, being at rest, it had received as much force in the opposite direction. If a man walking rapidly, or running, encounters another standing still, he suffers as much from the collision as the

* Quantitas motus quæ colligitur capiendo summam motuum factorum ad eandem partem, et differentiam factorum ad contrarias, non mutatur ab actione corporum inter se. (Principia, Lex III. cor. 3.)

† The pound of lead does indeed attract the earth only half as much as two pounds would do; nor does it *receive* from the earth but half as much; the power of attracting and of being attracted is the same.

man against whom he strikes. When two bodies moving in opposite directions meet, each body sustains as great a shock as if, being at rest, it had been struck by the other body with the united forces of the two. For this reason, two persons walking in opposite directions, receive from their encounter a more violent shock than might be expected. If they be of nearly equal weight, and one be walking at the rate of three and the other four miles an hour, each sustains the same shock as if he had been at rest, and struck by the other running at the rate of seven miles an hour.*

This principle accounts for the destructive effects arising from ships running foul of each other at sea. If two ships of 500 tons burthen encounter each other, sailing at ten knots an hour, each sustains the shock, which being at rest it would receive from a vessel of 1000 tons burthen sailing ten knots an hour. It is a mistake to suppose that when a large and a small body meet, the small body suffers a greater shock than the large one. The shock which they sustain must be the same; but the large body may be better able to bear it. When the fist of a pugilist strikes the body of his antagonist, it sustains as great a shock as it gives; but the part being more fitted to endure the blow, the injury and pain are inflicted on his opponent. This is not the case however when fist meets fist. Then the parts in collision are equally sensitive and vulnerable, and the effect is aggravated by both having approached each other with great force. The effect of the blow is the same as if one fist, being held at rest, were struck by the other with the combined force of both.†

238. The question may be asked, Why are the effects so much more severe when we fall from an eminence upon a naked rock than upon a bed of down? In both instances our fall is arrested, and we sustain a contrary and equal reaction; yet in the one case we might suffer hardly any injury, while in the other we should be bruised to death. The reason of the difference is this: when we fall on a bed of down, the resistance is applied gradually; when we fall on a rock, it is applied instantaneously. We do not strike the bed with the same force that we do the rock; we move along with the bed, and of course do not lose our motion at once, and we receive in the opposite direction merely what we lose. A violent blow, if equally diffused over

* Lardner's Mech. p. 47. † Ib. p. 48.

the human body may be sustained without injury. Thus, if an anvil be laid on the breast, a man may receive on it a heavy blow with a great hammer with impunity.*

239. There are many instances where action and reaction mutually destroy each other and no motion results. Thus, when a child stands in a boat and pulls by a rope attached to the stern, he labors in vain to make the boat advance. Dr. Arnott tells us of a man who attached a large bellows to the hinder part of his boat, with the view of manufacturing a breeze for himself, being ignorant that the reaction would carry the boat backward as much as the impulse of the artificial wind carried it forward.† A force which begins and ends within a machine has no power to move it.—(Emerson, 171.)‡

240. VARIABLE MOTION.—When a moving body is subjected to the energy of a force which acts on it without interruption, but in a different manner at each instant, the motion is called in general, *variable motion.*§ We have instances of variable motions in the unbending of springs, in the action of the wind on the sails of a ship, and in the action of gun powder on a ball while it is passing through the barrel of the gun. In each of these cases, the velocity of the moving body is constantly augmented, yet the degree of augmentation is diminishing until it finally ceases.

241. When a moving body receives, each successive instant, the same increase of velocity, it is said to be *uniformly accelerated.* If a small wheel were revolving without resistance, and, at the end of every second, I should apply a given impulse, the wheel would be

* Arnott's El. Phys. p. 104. † Ib. p. 107.

‡ It is common in elementary works on Mechanics, to find under the head of "reaction," a class of phenomena which evidently belong to a cause distinct from that of the mutual action of bodies. For example, a little steam carriage is sometimes exhibited, from which a jet of steam issues, and the carriage moves in the opposite direction. This, it is said, is owing to the reaction of the air upon the steam, being supposed analogous to the flying of a bird which beats the air with its wings, and is borne along by its reaction; but the motion of the carriage, in the foregoing experiment, is owing to a very different cause. Before the jet was opened the steam pressed equally on all sides of the vessel; as soon as the opening is made on one side, the pressure is removed from that side, but remains on the opposite side, and therefore gives motion to the vessel in that direction.

§ Gregory's Mech. I. 181.

uniformly accelerated; for, by its own inertia, it would retain all its previous motion, and, by the second law of motion, the repetition of the same force, at equal intervals, would increase its velocity at a uniform rate. If the intervals at which this force was repeated were indefinitely diminished, the same kind of effect would take place; and the same would evidently be the case, were the force to operate without cessation. Such a force is that of GRAVITY.

242. It has already been shown, in articles 4. and 7., that Gravity is a quality which belongs alike to all matter in proportion to its quantity; and that, at different distances from the center of the earth, it varies inversely as the square of the distance. The manner in which this force decreases as the distance increases will be seen at one view by the following table, beginning with the distance of the surface from the center.

Distance,	1	2	3	4	5	10	20	60
Attraction,	1	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$	$\frac{1}{100}$	$\frac{1}{400}$	$\frac{1}{3600}$

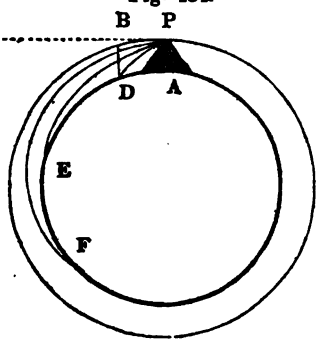
Hence it appears that a body placed 20 times as far from the center of the earth as the surface is from the center, is attracted only $\frac{1}{400}$th part as much; and at the distance of 60 times the radius of the earth the same force is diminished 3600 times.* At this distance therefore it would take 60 seconds, or one minute, for a body to fall through the space it falls at the surface of the earth in one second; that is, through $16\frac{1}{12}$ feet. But all distances within a few hundred feet of the earth bear so small a ratio to the earth's radius, that the force of gravity may be considered as the same unvarying force, in relation both to the weights of bodies and to the velocities with which they fall. (See Art. 8.)

243. It is not alone by the direct fall of bodies that the gravitation of the earth is manifested. The curvilinear motion of bodies projected in directions different from the perpendicular, is a combination of the effects of the uniform velocity which has been given to the projectile by the impulse which it has received, and the accelerated

* This last is nearly the distance of the moon from the earth; and it is found by calculation that the moon is actually drawn towards the earth, away from the straight line in which she tends to move, by exactly this force.

or retarded velocity which it receives from the earth's attraction. Suppose a body placed at any point P above the surface of the earth, and let PA be the direction of the earth's center. If the body were allowed to move without receiving any impulse, it would descend to the earth in the direction PA with an accelerated motion. But suppose that at the moment of its departure from P, it receives an impulse in the direction PB, which would carry it to B in the time the body would fall from P to A, then, by the composition of motion, (Art. 41.) the body must at the end of that time, be found in the line BD, parallel to PA. If the motion in the direction of PA were uniform, the body P would in this case move in the straight line PD. But this is not the case. The velocity of the body in the direction PA is at first so small as to produce very little deflection of its motion from the line PB. As the velocity, however,

Fig. 134.



increases, this deflection increases, so that it moves from P to D in a curve, which is convex towards PB. The greater the velocity of the projectile in the direction PB, the greater sweep the curve will take. Thus it will successively take the forms PD, PE, PF, &c. and that velocity can be computed,* which (setting aside the resistance of the air) would cause the projectile to go completely round the earth, and return to the point P from which it departed. In

* A cannon ball shot horizontally from the top of a lofty mountain, would go three or four miles. If there were no atmosphere to resist its motion, the same original velocity would carry it thirty or forty miles before it fell; and if it could be despatched with about ten times the velocity of a cannon shot, the centrifugal force would exactly balance the force of gravity, and the ball would go quite round the earth. (Arnott, El. Phys. p. 91.)

this case the body P would continue to revolve round the earth like the moon.

244. Hence it is obvious, that the phenomenon of the revolution of the moon round the earth, is nothing more than the combined effects of the earth's attraction, and the impulse which it received when launched into space by the hand of its Creator.* And were any of the heavenly bodies to explode, we may conceive that the fragments would proceed in a rectilineal direction until, approaching, severally, within the sphere of influence of some large body, whose attraction would combine with their projectile force, they would forever afterwards continue to revolve around that body, as the satellites revolve around their primaries.†

245. But the attraction of gravitation is manifested by comparatively small masses of matter. The effect of a high mountain is perceptible upon a plumb line, causing it to deviate sensibly from a perpendicular, so that the same star in the zenith would change its apparent place when viewed on opposite sides of the mountain. This was observed by two French astronomers, near Mount Chimborazo in South America, as early as the year 1738; and the experiment was repeated in 1772 with all possible accuracy, by Dr. Maskelyne, astronomer royal of Great Britain, at the base of the mountain Schehallien, in the eastern part of Scotland.‡ Mr. Cavendish, a distinguished English philosopher of the last century, rendered sensible even the attraction of a sphere of lead, by bringing it near a small bullet, suspended from one arm of an exceedingly delicate balance. The sphere when brought near the bullet disturbed its equilibrium.§

246. By gravity, bodies are directed towards the center of the earth. We are not to infer from this fact that there is any peculiar force, (like that of a large magnet for example,) residing at the center, but merely that the effect of the earth, taken as a whole, is the

* Lardner's Mechanics, p. 79.

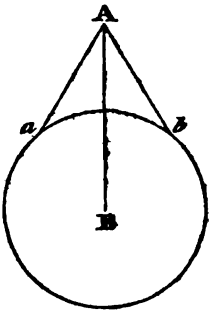
† This has actually been supposed of the four new planets, Ceres, Pallas, Juno and Vesta.

‡ Ed. Encyclopedia, III. 76.

§ Ib. Vol. III. Lardner's Mech. 82.

same as though its matter were condensed into the center. The line of attraction passes through the center because such a line is the *resultant* of the separate attractions of all the particles composing a sphere. Thus if AB be a line passing through the center of the sphere, and A be any body, and *a* and *b* two particles of matter in the sphere equally distant from A, it is evident that their combined actions would be expressed by the diagonal which would coincide in direction with AB. The same is true of any other particles taken equally distant from A in opposite hemispheres. At different parts of the earth, therefore, the directions of falling bodies are not *parallel*, but form converging lines. But on account of the great magnitude of the earth, two places 100 feet distant will not vary *one second*, and when a mile asunder, they will not differ *one minute* from perfect parallelism.* For all the purposes of machinery, therefore, as well as for experiment, the direction of the lines of gravity may be considered as parallel.

Fig. 135.



247. Since bodies in falling towards the earth are uniformly accelerated, the *velocity* acquired must be proportioned to the time the body has been falling: at the end of ten seconds it has acquired ten times the velocity which it had at the end of one second. And in Art. 29, it has been shown that the *spaces* described are proportioned to the squares of the times; so that the space described during 100 seconds, is not merely 100 times as great as that described in one second, but it is the square of 100, or 10,000 times as great. This conclusion was arrived at mathematically, long before it was established by actual experiment. There were two difficulties which stood in the way of such a verification, viz. the little time occupied in descending through such perpendicular heights as the experimenter can command, and the resistance of the air, which, when the velocity becomes great, acts as a powerfully retarding force. We can rarely command a perpendicular eminence of more than four hundred feet, and yet the time occupied in the whole descent is only about five seconds, a period too short to enable us to mark distinctly

* Gregory's Mech. I. 193.

the respective rates at which the successive intervals are described. Atwood's Machine (Fig. 132. p. 193.) affords the means of obviating both these difficulties, and verifying the laws of falling bodies with great accuracy.

248. The object of the machine, so far as respects experiments on falling bodies, is to render the descent of bodies so *gradual*, that the relation between the times and spaces, can be accurately observed. By recurrence to the figure, and to the descriptions given in Art. 223, we shall readily see how this object may be accomplished. The weights D and E each equal $31\frac{1}{2}$ ounces, and of course the quantity of matter in both is 63 ounces. Now, since one of these rises as the other descends, the force of gravity retards the one as much as it accelerates the other, and they are in effect the same as though they were entirely destitute of gravity. If a small weight, as one ounce, were let fall *freely* from the top of the machine, it would fall through so small a space almost in an instant, and we should be unable to mark the rate at which it would pass over the successive portions of the graduated scale FG; but if it be laid on the weight D, it must carry D along with it; that is, it must make D descend and E ascend, and therefore the motion belonging to one ounce, will be distributed throughout 64 ounces, and its velocity will be retarded in the same ratio. Consequently, the weight D will descend only $\frac{1}{64}$th part as fast as a body falling freely; and as a body falling freely descends in one second about 16 feet or 192 inches, the weight D will descend $\frac{192}{64} = 3$ inches in one second. The comparative progress of this weight, and of a body falling freely for several successive seconds, will be seen in the following table.

Time,	1	2	3	4	5	6
Body falling freely, in feet,	$16\frac{1}{12}$	$64\frac{1}{3}$	$144\frac{3}{4}$	$257\frac{1}{3}$	$402\frac{1}{12}$	579
Do. in Atwood's Machine, in inches,	3	12	27	48	75	108

Hence it appears that in six seconds, while a body would fall freely through 579 feet, it would in the same time descend only nine feet in Atwood's Machine. But the latter is a uniformly accelerated velocity, and subject to the same laws as the former, and it may therefore be employed to investigate the laws of falling bodies. The results correspond remarkably with theory, so that when the instru-

ment is well constructed, and managed skilfully, the descending weight clicks upon the stage or brass plate K, at the very instant required.

249. We see in nature the law of acceleration of falling bodies indicated, by the impetuosity with which bodies fall from any considerable height upon the earth. Meteoric stones, falling from the sky, sometimes bury themselves deep in the ground. Aeronauts that have fallen from balloons have been dashed in pieces.* It is, however, a rare occurrence to see a body falling from any great height *perpendicularly:* most instances of accelerated motion which come under our observation are in bodies falling down *inclined planes*, where the same law of acceleration prevails. (Art. 158.) A fragment of rock descending from the side of a mountain, has its speed augmented as it goes, until its momentum becomes irresistible, and large trees are prostrated before it.

250. A very remarkable example of the acceleration of bodies descending down inclined planes, occurs at the *Slide of Alpnach* in Switzerland. On Mount Pilatus, near Lake Luzerne, is a valuable growth of fir trees, which on account of the inaccessible nature of the mountain, had remained for ages undisturbed, until within a few years, a German engineer contrived to construct a trough in the form of an inclined plane, by which these trees are made to descend by their own weight, through a space of eight or nine miles, from the side of the mountain to the margin of the lake. Although the average declivity is no more than about one foot in seventeen, and the route often circuitous and sometimes horizontal, yet so great is

* Any liquid falling from a reservoir, forms a descending mass or stream, of which the bulk diminishes from above downwards, in the same proportion in which the velocity increases. This truth is well exemplified by the pouring out of molasses or thick syrup: if the height of the fall be considerable, the bulky mass which first escapes, is reduced, before it reaches the bottom, to a small thread; but the thread is moving with proportionally greater speed, for it fills the receiving vessel with great rapidity. The same truth is exhibited on a grand scale in the Falls of Niagara, where the broad river is seen first bending over the precipice, a vast slow moving mass, then becoming a thinner and thinner sheet; until it flashes into the deep below, almost with the velocity of lightning. (Arnott's El. Phys. 79.)

the acceleration, that a tree descends the whole distance in the short space of six minutes. To a spectator standing by the side of the trough, at first is heard, on the approach of the tree, a roaring noise, becoming louder and louder; the tree comes in sight at the distance of half a mile, and in an instant afterwards shoots past with the noise of thunder and the rapidity of lightning. When a tree happens to "bolt" from the trough, it cuts the standing trees quite off.* (See pp. 145, 146. Prob. 5. and 6.)

251. COMPOSITION AND RESOLUTION OF MOTION.—*Simple* motion is that which arises from the action of a *single* force; *Compound* motion is that which is produced by *several* forces acting in different directions. Strictly speaking, we have no example of a simple motion, since in the absolute motion of all bodies, their own proper motion is combined with that of the earth in its diurnal and annual revolutions, and we know not with how many others. (Art. 215.) In an enlarged sense, therefore, all motions are compound. But in the foregoing distinctions we have reference only to relative motions, as those which take place among bodies on the earth. In accordance with the second law of motion, (Art. 231.) a force striking upon a body in motion, will produce the same change of motion as though the body had been at rest when the force struck it. This may at first view appear inconsistent with experience, especially in regard to opposite motions. Let us, therefore, consider the principle in its application to several different cases. Conceive the ice of a frozen river to be first stationary, and afterwards to float down with the current. Standing on the bank I roll a ball directly across the river. Will it pass, in the same direction in both cases? It will not; for in the first case it will pass across *perpendicularly* to the banks, and in the second case it will go across *diagonally*. But now let me stand upon the ice and roll the ball. Since I float along with the ice I am at rest with respect to that motion, and the ball, though moving diagonally as before, appears to me to go directly across the stream.† If the ball was rolled not directly across but obliquely, making any angle with the bank, if I stood upon the floating ice, and was there-

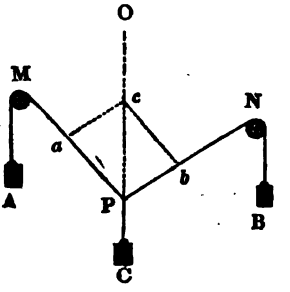
* Playfair's Works, 1. 96.

† In the same manner two persons sitting in a boat under sail, toss a ball from one to the other in the same manner as though they were at rest on land.

fore at rest with respect to one of the motions of the ball, I should see the other motion in the same manner as though I had stood on the shore and the ice had been at rest. But if when a ball is rolling towards me, and I strike it in a direction exactly opposite to its course, but do not stop it, can the blow be said to produce the same change of motion as though the body had been at rest when the blow was applied? Ans. If I had been moving in the same manner as the ball before the blow, then stopping a part of the motion of the ball, would have given it a relative motion in the opposite direction; since having none of my own motion stopped, I should leave it behind. This is what takes place when a cannon ball is fired in a direction contrary to that in which the earth is revolving about the sun. The cannon moves onward and leaves the ball behind. (Art. 215.)

252. The laws respecting the composition and resolution of motion which are demonstrated in Chap. III. p. 33,, admit of being satisfactorily confirmed by experiment. Let two small wheels M, N, be attached to a wall or board. Let a thread be passed over them, having weights A and B, hooked upon loops at its extremities. From any part P of the thread, let a weight C be suspended, in such a manner as to be in equilibrio with A and B. The weight C, therefore, is the resultant of the forces A and B; and since its direction is that of gravity, it will be represented by a line drawn directly upwards from P. From P, on the line PO, take P*c* having as many inches as

Fig. 134.



there are ounces in C; and from *c* draw *ca* parallel to NP and *cb* parallel to PM. If the sides P*a* and P*b* of the parallelogram thus formed be measured, it will be found that P*a* will consist of as many inches as there are ounces in A, and P*b* of as many inches as there are ounces in B; consequently, the lines P*c*, P*a*, and P*b*, have the

same ratio to each other as the forces or weights C, A, and B. But P*c* is the diagonal, and P*a* and P*b* are the sides of a parallelogram. Hence the diagonal of a parallelogram represents a force equivalent to the two forces represented by the sides.*

This experiment is an illustration of the composition of *forces*, rather than of the composition of *motions;* but a simple experiment will show that the same law holds good with respect to a body actually set in motion by two different forces. A ball is placed at one of the corners of a smooth table. To the same corner are attached two springs, respectively in the line of the two sides of the table, and capable of giving a simultaneous impulse to the ball. The springs moreover are so proportioned to each other, that one will drive the ball across one side of the table, in the same time that the other will drive the ball across the other side. Now on letting go both springs at once, the ball will pass, in the same time, across the diagonal of the table to the opposite corner.†

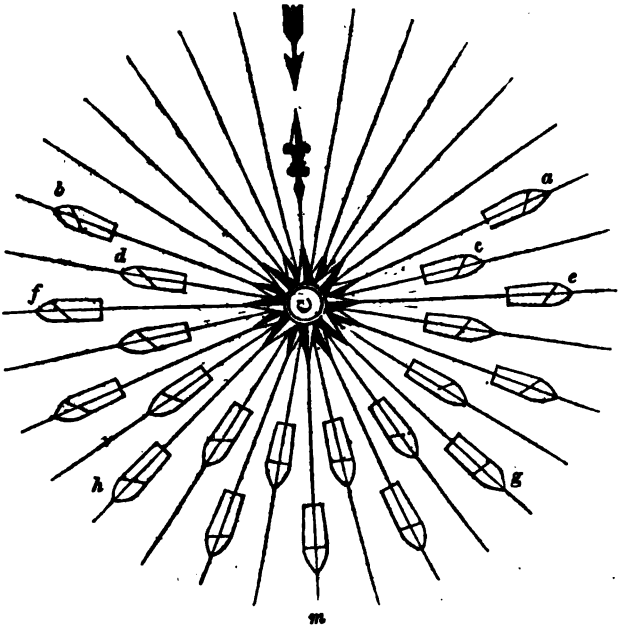
253. We daily observe examples strikingly illustrative of these laws. In crossing a river the boatman heads up the stream, and so combines the direction of the boat with that of the current, as to move directly across in a line, which is the diagonal between the two. Rowing, swimming, and flying, are severally instances of motion in the diagonal between two forces. In feats of horsemanship, when the rider leaps up from his saddle, we are surprised not to see the horse pass from under him; but he retains the motion he has in common with the horse, and does not in fact ascend perpendicularly, but obliquely, rising in one diagonal, and falling in another. In the common feats of jumping through a hoop, and alighting again on the saddle, an inexperienced rider would be likely to project his body forward in the same manner as he would do in leaping through the same hoop from the ground. In such a case, instead of alighting on the saddle, he would alight either before the horse or upon his head or neck. All that is requisite in order to execute this feat, is to leap directly upwards from the saddle to a sufficient height to clear the lower part of the hoop with the feet. By the speed which the rider has in common with the horse, his body will, without any exer-

* Lardner's Mech. p. 51. † Lib. Useful Knowledge, Mechanics, p. 5.

tion on his part, pass through the hoop, and he will alight again on the saddle, on the other side, in his descent.*

254. The *sailing of a ship* affords an instructive illustration of the principles of the composition and resolution of motion. When a ship sails in the same direction as the wind, she is said to be *scudding*, or sailing *before the wind*, and if she had but one sail, it would act with the greatest advantage, when perpendicular or nearly so to

Fig. 135.



the wind. When a ship advances against the wind, and endeavors to proceed in the nearest direction possible to the point of compass from which the wind blows, she is said to be *close hauled*. A large ship will sail against the wind with her keel at an angle of six points with the direction of the wind, and sloops and smaller vessels may sail much nearer. When a ship is neither sailing before the wind, nor is close hauled, she is said to be *sailing large*. In this case, her

* Lib. Useful Knowledge, Mech. p. 5.

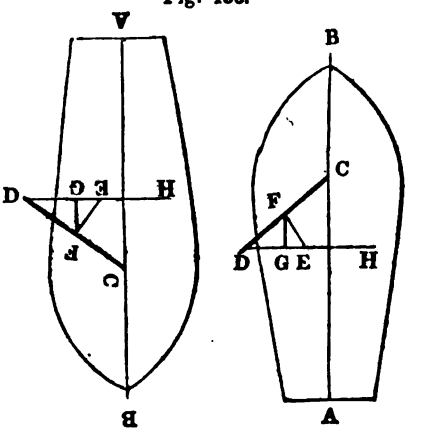
sails are set in an oblique position, between the direction of the wind, and that of the intended course; as represented in the various plans of vessels in fig. 135, where the direction of the wind is represented by the arrow, and the position of the yards and sails, which are necessary for proceeding on the various points of compass, are shown by the transverse lines on each plan. The relation of the wind to the course of the vessel is determined by the number of points of the compass between the course she is steering, and the course she would be steering if close hauled. In fig. 135, the ships *a* and *b* are close hauled, and the ships *c* and *d* (the former steering east by north, and the latter west by north) have the wind one point large. The ships *e* and *f*, one steering east and the other west, have the wind two points large. In this case, the wind is at right angles with the keel, and is said to be *upon the beam*. The ships *g* and *h* steering southeast and southwest, have the wind six points large, or, as it is commonly termed, *upon the quarter*, and this is considered as a very favorable manner of sailing, because all the sails cooperate to increase the ship's velocity, whereas when the wind is directly aft, as in the vessel *m*, it is partly intercepted by the after sails, and prevented from striking with its full force on those which are forward.*

255. To one who has never studied the doctrine of the composition and resolution of forces, it is apt to appear mysterious that a ship is able to sail with a wind partly ahead, and still more that two ships are able to sail in exactly opposite directions by the same wind. It is proposed to explain these phenomena. Let AB (fig. 136.) represent the keel of a ship, and CD the sail; and let the wind come in from the side, in the direction of HD. Let DE represent the whole force of the wind, and resolve DE into two forces, viz. into EF perpendicular, and FD parallel to the sail DC. Then it is manifest EF alone represents the effective force of the wind upon the sail. But EF is not wholly employed in urging the ship forward, since it is oblique to her course; therefore, again resolve EF into FG parallel with the course and GE at right angles with it. The latter force is lost by the lateral resistance of the water, while FG is employed in propelling the ship on her way.

* Bigelow's Elements of Technology, p. 218.

By inspecting Fig. 136. it will readily be seen that another ship may sail in the opposite direction by the same wind. When the wind strikes the sail at right angles, or in the direction EF, then only

Fig. 136.



one resolution is necessary; for if EF represents the whole force of the wind, FG will represent the force that propels the ship forward, while GE will represent the part which is lost by the lateral resistance of the water.

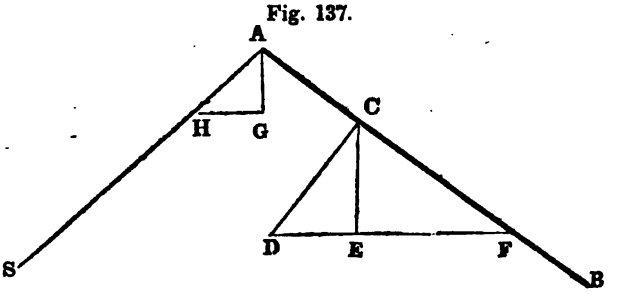
Since, resolving the force of the wind after the foregoing manner, the effective part of the force, viz. FG, will not wholly disappear until the wind is directly ahead, it might seem possible to sail much nearer the wind than is found to be actually practicable. But, though, on account of the peculiar shape of vessels, the forward resistance is much less than the lateral, yet it is *something*, and therefore requires more or less of the force that acts parallel with the keel to overcome it.

256. The doctrine of the composition and resolution of motion, by reducing a great number of complicated motions to one, or by enabling us to estimate the precise influence of forces that act obliquely, has greatly simplified inquiries in Mechanics, and proportionally advanced the science. It is only by such means that the complex motions of the heavenly bodies, and the equally diverse forces that control them, could ever have been understood. The subject has also an extensive and important application in estimating the powers

of machinery. Since, for example, any one side of a triangle or polygon, is always less than the sum of all the remaining sides, it follows that a mechanical effect will always be more economically produced by a single force acting in the proper direction, than by a number of forces acting in different directions.

257. By Art. 55. it appears, that *a body acted upon at the same time by three forces, represented in quantity and direction by the three sides of a triangle taken in order, will remain at rest.* A kite at rest in the air, is commonly mentioned as an example of this, the three forces being, the direction of the wind, the weight of the kite, and the action of the string. Let AB be a kite, held by the string AS. Let DF represent the force of the wind blowing horizontally, and resolve it into two forces, viz. DC perpendicular, and CF parallel to the kite. Then DC will be the only effective part of the wind, since that part which acts parallel to the kite, can have no in-

Fig. 137.



fluence on its motions. Again, resolve CD into two forces, namely, CE perpendicular, and DE parallel to the horizon. Then CE will represent the upward force of the wind, and DE its force in a horizontal direction. Now when the string AS makes such an angle with the kite that its downward force AG, added to the weight of the kite, shall equal CE, and its horizontal force HG, equal DE, the kite will be at rest.

258. When two motions which are not in the same straight line, are combined, one of which is uniform, and the other accelerated, the moving body describes a *curve*. (Art. 48.) If two accelerating forces act, in which the rate of acceleration is the same, the motion is *rectilinear;* but if the rate of acceleration is different, the motion is still *curvilinear*.

CHAPTER II.

OF THE CENTER OF GRAVITY.

259. The principles which have been discovered respecting the composition and resolution of forces, and respecting the center of gravity, have alike contributed greatly to simplify the doctrines of Mechanics. It is characteristic of a great and penetrating mind, to devise means of divesting intricate subjects of their complexity, and thus to bring easily within the grasp of the mind, subjects otherwise too much involved to be within its comprehension. By the rule of simple multiplication, we easily multiply any number by one thousand: indeed, it is nothing more than to annex three cyphers to the number itself; but how tedious would be this process, were the rule of multiplication undiscovered, and we were unacquainted with any other method of arriving at the result, except to add the given number to itself one thousand times. In like manner by means of the rules for the composition of motion, we are enabled to reduce a thousand different motions to one; and by the doctrine of the center of gravity, we are taught how we may make a force, situated at one single point, equivalent to an infinite number of forces, situated in as many different points. And, instead of pursuing the endless diversities of motions to which the different parts of a complicated system of bodies may be subject, we are taught how to follow merely the motions of a single individual point. By the earth's attraction, all the particles which compose the mass of a body, are solicited by equal forces in parallel directions downwards. If these component particles were placed in mere juxtaposition, without any mechanical connexion, the force impressed on any one of them, could in nowise affect the others, and the mass would, in such a case, be contemplated as an aggregation of small particles of matter, each urged by an independent force. Then, according to Art. 60, the *resultant* constitutes another force parallel to the others. But the bodies which are the subjects of investigation in mechanical science, are not found in this state. Solid bodies are coherent masses, the particles of which are firmly bound together, so that any force which affects one,

being modified according to circumstances, will be transmitted through the whole body.*

260. As all bodies which are subjects of mechanical inquiry, on the surface of the earth, must be continually influenced by terrestrial gravity, it is desirable to obtain some easy and summary method of estimating the effects of this force. To consider it, as is unavoidable in the first instance, the combined action of an infinite number of equal and parallel forces, soliciting the elementary molecules downwards, would be attended with manifest inconvenience. An infinite number of forces, and an infinite subdivision of the mass, would form parts of every mechanical problem. To overcome this difficulty, and to obtain all the ease and simplicity which can be desired in elementary investigations, it is only necessary to determine some force, whose single effect shall be equivalent to the combined effect of the gravitation of all the molecules of the body.†

Such a force is obtained by supposing all the action to be concentrated in the center of gravity. We have already defined it thus: *the center of gravity of a body, is that point about which, if supported, all the parts of a body (acted upon only by the force of gravity,) would balance each other in any position.*‡ (Art. 64.)

261. *To find the center of gravity by experiment,* several different methods present themselves. We will first suppose the body to be in the shape of a piece of board, of uniform thickness. Suspend it by one corner, and from the same corner let fall a plumb line, and mark its line of direction on the surface of the board. Suspend the board from any other point, and mark the line of direc-

* Lardner's Elements of Mechanics, p. 107. † Ib. p. 108.

‡ Others define it to be "the point through which passes the resultant of all the particular forces exerted by the gravity of the several parts of the body, or system of bodies, in whatever position the body or system is placed." (Cambridge Mech. p. 45.)

"The resultant of any number of parallel forces continues of the same intensity, and passes through the same point, whatever be the direction of the forces; hence it is called the Center of Parallel Forces. When the body is moving forward in a straight line, under the action of forces other than gravity, it is called the *center of Inertia;* when the body is acted upon by gravity, it is called the center of gravity." (Renwick on the Steam Engine, p. 14.)

tion of the plumb line as before, and the point where these lines intersect each other, must obviously be the center of gravity, since that center is in both of the lines. (Art. 65.)

But when the body is not of uniform thickness, but is any irregular solid, suspend the body by a thread, and let a small hole be bored through it, in the exact direction of the thread, so that if the thread were continued below the point where it is attached to the body, it would pass through this hole. The body being successively suspended by several different points in its surface, let as many small holes be bored through it in the same manner. If the body be then cut through, so as to discover the directions which the several holes have taken, they will all be found to cross each other at one point within the body. Or the same fact may be discovered thus: a thin wire which nearly fills the holes being passed through any one of them, it will be found to intercept the passage of a similar wire through any other.*

A convenient method of finding the center of gravity of a body is, *to balance it in different positions across a thin edge*, as the edge of a knife or a prism. The same thing may be effected, when the shape of the body will admit of it, by laying it on the edge of a table, and letting so much of it project over the edge, that the slightest disturbance will cause it to fall. The center of gravity is the point in which the several lines marked on the body, where the edge cuts it, intersect one another. From some or all of the foregoing trials, the center of gravity of bodies may be nearly ascertained; but in order to find it with absolute exactness, we are frequently obliged to resort to intricate fluxionary processes.

Properties of the center of gravity.

262. By whatever method the center of gravity of a body has been ascertained, we shall find that when that is supported, the body will remain at rest in every position. Thus a globe will stand securely on a very small perpendicular support, since that support will necessarily be under the center of gravity; a lever, as the beam of a balance, poised on its center of gravity, will be at rest in every position it takes while turning round the fulcrum, and however irreg-

* Lardner's Mech. p. 110.

ular the body may be, it will, when balanced on its center of gravity, obstinately maintain its position.

When a body is suspended by an inflexible rod from a center of motion, and revolves around it, it will be at rest only *when the center of gravity is either directly below, or directly above the center of motion.* For it is only in these two cases, that the center of gravity will be in the line which is drawn through the center of motion perpendicular to the horizon. The stationary point above the center of motion is very *unstable*, since the slightest disturbing force throws the body out of the line of direction, when, by the force of gravity, it immediately descends to the lowest point it can reach, and vibrates about that point until it finally settles itself with the center of gravity immediately under the point of suspension; and whenever it is thrown out of this position, the same vibrations are renewed until it resumes it. When therefore the center of gravity is at the lowest point it is capable of reaching, the equilibrium is *stable*, since the body obstinately maintains that position. On this principle, gates which have their center of gravity raised as they are opened, shut spontaneously.

263. The stability of a body not only requires that the center of gravity should be low, but that *the line of direction* (or, the line which is drawn through the center of gravity perpendicular to the horizon) *should fall within the base.* (Art. 69.) The farther it falls from the extremity of the base, the more stable is the position. Hence the stability of a pyramid, when standing on its broad base, and its instability when inverted. For the same reason, all broad vessels, as steamboats, are difficult to upset, while vehicles with narrow bases are easily overturned. When a load is so situated as to raise the center of gravity, it increases the liability to upset, because it increases the facility with which the line of direction is thrown without the base. Thus carts loaded with hay, or bales of cotton, are very liable to be overturned. The same is true of stages carrying passengers or baggage on the top. On the other hand, a large ship well supplied with ballast is capsized with great difficulty, since the center of gravity of all parts of the ship is so low, as to render it difficult to throw the line of direction without the base. Yet if the center of gravity is very low, a ship will rock excess-

ively in a rough sea, since the upper parts near the deck, move over a greater space in proportion as their distance from the center of gravity is greater.

264. There are many remarkable structures which *lean* or incline a little; but so long as the line of direction falls within the base, and the parts of the mass have sufficient tenacity among themselves to hold together, the structure will stand. The famous tower of Pisa, was built intentionally inclining to frighten and surprise: with a height of one hundred and thirty feet, it overhangs its base sixteen feet.* This circumstance greatly enhances the emotion of the spectator from its summit. Many ancient spires and other tall structures, are found to have lost something of their perpendicularity.

Rocking stones are rocks which are sometimes found so exactly poised upon their center of gravity, that a very small force is sufficient to put them in motion.† The rocking of a balloon when it begins to ascend, affords an illustration of the tendency of bodies to vibrate around the center of gravity.

265. The *motions of animals* are regulated in conformity with the doctrines of the center of gravity. A body is seen tottering in proportion as it has great altitude and a narrow base; but it is a peculiarity in man to be able to support his figure with great firmness, on a very narrow base, and under constant changes of attitude. This faculty is acquired slowly, because of the difficulty. The great facility with which the young of quadrupeds walk, is ascribed in part to their broad supporting base. Many of our most common motions and attitudes, depend for their ease and gracefulness, upon a proper adjustment of the center of gravity. The erect posture of a man carrying a load upon his head—leaning to one side when a heavy weight is carried in the opposite hand—leaning forward when a weight is on the back—or backward when the weight is in the arms; these are severally examples in point. When a man rises from his chair, he brings one foot back, and leans the body forward, in order to bring the center of gravity over the base; and without adjusting

* Some travelers, however, are of opinion that the inclination of the tower of Pisa, is the effect of time. It is said that an ancient picture of the tower represents it as erect. Arnott's El. Phys. p. 121.

† Amer. Journal of Science, Vol. 7.

it in this manner, it is hardly possible to rise. A man standing with his heels close to a perpendicular wall, cannot bend forward sufficiently to pick up any object that lies on the ground near him, without himself falling forward.

The art of rope or wire dancing, depends in a great degree upon a skilful adjustment of the center of gravity. The rope dancer frequently carries in his hand a stick loaded with lead, which he so manages as to counterbalance the inclinations of his body which would throw the line of direction out of the base. Upon a similar principle the equestrian balances himself on one foot on a galloping horse.

The *vegetable creation* is subject also to these general laws of nature. Trees by the weight and height of their tops would seem peculiarly liable to fall; but their roots afford a corresponding breadth of base, while their perpendicular trunks, and the symmetrical disposition of the branches, conspire to increase their stability.

266. The position of the center of gravity of any number of separate bodies, is never altered by the *mutual action* of those bodies on each other. If, for example, two bodies, by mutual attraction, approach each other, the center of gravity remains at rest, until finally the bodies meet in this point. If, by their mutual action, they contribute to make each other revolve in orbits, it is around their common center of gravity. Thus the earth and moon revolve around a common center of gravity: the same is true of the sun and all the bodies that compose the solar system. Were the centrifugal force to be suspended, and the bodies abandoned to the mutual action of each other, they would all meet in their common center of gravity. (Art. 81.) This naturally results from the principle that the momenta on opposite sides of the center of gravity are equal, and that bodies by their mutual action produce equal momenta in each other.

Solution of Problems by means of the center of gravity.

267. The doctrines of the center of gravity suggest the readiest method of solving a great number of practical problems.

Suppose three persons were carrying a stick of timber, (A by himself supporting one end, and B and C by a handspike lifting

together towards the other end,) and it were required to determine at what distance from the end of the stick the handspike must be placed in order that the three persons might bear equally. A stick of timber being a body of regular shape and uniform density, has its center of gravity coincident with the center of magnitude. We may therefore proceed on the supposition that the entire weight is collected in the center. Now in order that B and C may together lift twice as much as A, they must be twice as near the center. But the distance of A from the center is one half the length of the stick; therefore the distance of the required point from the center is one fourth the length of the stick, and consequently it is one fourth the same length from the end of the stick.

The result thus obtained from theory, may be easily submitted to the test of experiment. For if we take the weight of the stick of timber with a pair of steelyards, and then, resting one end on some support, attach a cord at the distance of one fourth of the length of the stick from the other end, and thus connect the stick with the steelyards, we shall find the weight equal to two thirds of the whole.*

268. The method of finding *the areas of the surfaces and the solid contents of bodies*, particularly of such as are of unusual figure, may frequently be very much simplified, by applying the principles of the center of gravity. This is sometimes called the *centrobaryc* method. It was discovered by Pappus, an ancient mathematician of Alexandria, but was more completely discussed and illustrated by Guldinus, Professor of Mathematics at Rome, about the year 1640.† This remarkable property of the center of gravity is expressed in the following propositions:

1. *If any line whatsoever revolves about a fixed point, the* SURFACE *which it generates is equal to the product of the given line into the circumference described by its center of gravity.*

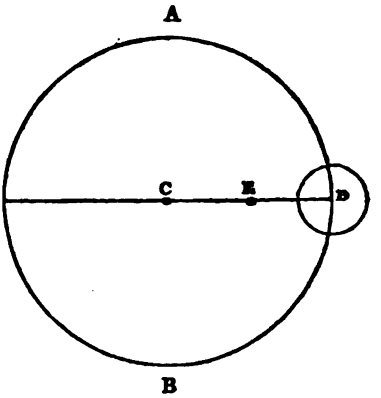
* This experiment may be repeated with much precision in the following manner. Take a Gunter's scale, well made, and ascertain its weight by a delicate balance. Let one end rest on a sharp edge, as the edge of a prism, and attaching a string at the distance of one fourth the length of the scale from the other end, connect it with one of the arms of the balance: the weight will be exactly two thirds of the whole.

† Hence these properties of the center of gravity are sometimes called *Guldinus' Properties.*

2. *If a surface having any figure whatsoever, revolves about an axis, the* SOLID *generated is equal to the product of that surface into the circumference described by its center of gravity.*

269. Thus, the straight line CD, (Fig. 137.) revolving about the center C, describes a circle whose surface is equal to CD into the circumference of the circle described by its center of gravity E. This is evident also from the consideration that, since E is the center of the line CD, the circumference described by it will be half the length of the circumference ADB; and the area of a circle is equal to the product of the radius into half the circumference.

Fig. 137.



Again, the small circle, having its center coincident with the extremity of the line D, and revolving round on the circumference of the circle described by CD, being every where perpendicular to the plane of the circle, would describe a solid figure like the ring of an anchor; and the line described by the center of gravity, that is, the circumference of the circle, multiplied into the area of the revolving figure, would give the solidity of the ring. In like manner, in a cone, the solidity is equal to the *area* of the generating triangle, multiplied into the circumference of the circle, formed by the revolution of the center of gravity of the triangle; and the surface is equal to the product of the *perimeter* of the same triangle, multiplied by the circumference described by the center of gravity of the same perimeter.*

* For demonstrations and farther illustrations of these propositions, see Appendix.

CHAPTER III.

OF MACHINERY.

270. The organs employed in communicating motion, are tools, machines, and engines. *Tools* are the simplest instruments of art; these when complicated in their structure, become *machines;* and machines when they act with great power, take the name of *engines.* Among the ancients, machines were confined chiefly to the purposes of *architecture* and *war;* and they were moved almost exclusively by the *strength of animals.* Thus, in building one of the great Pyramids of Egypt, vast masses of stone were raised to a great height, amounting together to 10,400,000 tons. In this labor were employed 100,000 men for twenty years. The advantage which man has gained by pressing into his service the great powers of nature, instead of depending on his own feeble arm, is evinced by the fact, that by the aid of the steam engine, one man can now accomplish as much labor as 27,000 Egyptians, working at the rate at which they built the Pyramids.* In war also, while the use of gunpowder was unknown, engines of great power were invented for throwing stones and javelins, and for demolishing fortifications. Such were the Catapulta, the Balista, and the Battering Ram, of the Romans. Yet it is remarkable, that during many ages, while such powerful auxiliaries were employed in architecture and in war, the ancients should have made so little use as they did of machinery in the ordinary processes of the arts. The practice of grinding corn by hand, which was chiefly performed by women, was prevalent at Rome until the time of Augustus, when we find the first mention made of water mills.†

271. The *Mechanical Powers,* being the principal instruments of art, will first require our attention. They have already been considered *theoretically;* we are now to consider them *practically.* It will be recollected that, when two forces act on one another by means of any machine, that which gives motion is called the *power;* that

* Dupin. † Ed. Encyc., Art. *Mechanics.*

which receives motion, the *weight*. (Art. 97.) The weight includes not only the proper weight of the body or bodies moved, but also every kind of resistance opposed to the action of the power, whether it arises from the quantity of matter in the body moved, or from the inertia of the machine itself, or from the air, or from friction. It will be farther recollected, that an equilibrium is produced between two forces, when their *momenta* are equal. (Art. 149.) Now, since momentum is compounded of quantity of matter and velocity, a given momentum may be produced, either by giving a great velocity to a small weight, or a small velocity to a great weight. The consideration of this subject will be resumed hereafter.

THE LEVER.

272. The principle of the lever has a most extensive application in the arts, and the forms under which it occurs are very various. We may contemplate it as having equal or unequal arms.

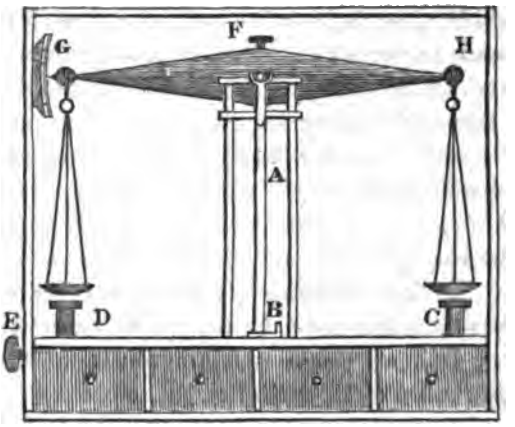
The *balance* affords the most common example of a lever with equal arms. The necessity of arriving at the weight of bodies with the greatest degree of accuracy in pecuniary transactions, and more especially in delicate scientific researches, as those of chemical analysis, has induced men of science, and artists, to bestow great and united attention upon the construction of this instrument, until they have brought it to an astonishing degree of perfection.

273. The principal parts of the balance are the beam GH (Fig. 138.) the points of suspension G and H, and the fulcrum F. In order to construct a perfect balance, the most important particulars to be attended to, are the length of the arms, that is, of the beam; the situation of the center of gravity of the whole instrument, with respect to the fulcrum or center of motion; and the position of the point of suspension.

(1.) The sensibility of the balance is increased by increasing the *lengths of the arms*; but unless the arms, when long, are at the same time of considerable weight, they will not have the requisite strength, but will be liable to bend; and an increase of weight, adds to the amount of friction on the center of motion. It is not common, therefore, to make the arms of a very delicate balance more than nine

inches in length; and, for the purpose of uniting lightness with strength, the beam is composed of two hollow cones placed base to base, as in Fig. 138.

Fig. 138.



(2.) The *center of gravity* of the instrument, must be a little below the center of motion. For if the beam is balanced on its center of gravity, it will remain at rest in every position, (Art. 64.) whereas it must be at rest, only when in a horizontal position. If the center of gravity is above the center of motion, the position is too unstable, (Art. 262.) and on the least disturbance of the equilibrium, the beam will be liable to upset. Finally, if the center of gravity is too far below the center of motion, the equilibrium will be too stable. Hence, in very delicate balances, the center of motion is placed a little above the center of gravity.

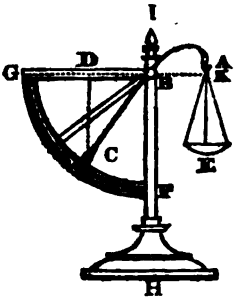
(3.) The *points of suspension* must be in the same right line with the center of motion. For since when weights are added to the scales, the effect is the same as though they were concentrated in the points of suspension, (Art. 99.) were those points above the center of motion, the center of gravity would be liable to be shifted above the center of motion, when the beam would upset; and if the same points were below the center of motion, unless the weights added were large, the center of gravity would be too low, and the equilibrium too stable.

274. In order to prevent friction as much as possible, the *fulcrum* is made of hardened steel, and shaped into a triangular prism, or knife edge, smoothly rounded, and turning on a plane of agate or steel, or some other very hard and polished substance.

It is only by a nice attention to all these particulars, that artists have been able to give to the balance so great a sensibility. Some have been made to turn with the 1000th part of a grain.* By loading the beam, the sensibility of the instrument is diminished; it is customary, therefore, to estimate its power, by finding what part of the weight with which it is loaded it takes to turn it. Thus, if when loaded with 7000 grains, it will turn with one grain, its power is $\frac{1}{7000}$. A balance constructed by Ramsden, a celebrated English artist, for the Royal Society, turned with the *ten millionth* part of the weight.† Delicate balances are usually covered with a glass case to prevent agitation from the air, and to secure them from injury. Figure 138, represents an instrument of this kind made for the Royal Institution of Great Britain.‡

275. The *bent lever balance* is represented in figure 139. The weight C acts as though it were concentrated in the point D, (Art.

Fig. 139.



104.) and the weight in the scale acts at K; hence an equilibrium will take place, when the article weighed has to C the same ratio as DB has to BK. Now every increase of weight added to the

* Nicholson's Chemical Dict.—Kater in Lardner's Mechanics.

† Young's Lectures on Nat. Phil. I. 125.

‡ For a full account of the most accurate balances, see Kater on "Balances and Pendulums" in Lardner's Mechanics, p. 278.

scales causes C to rise on the arc FG, and D to recede from B. Hence the different positions of C, according as different weights are added to the scale, may be easily determined, and the corresponding numbers marked on the scale FG.

276. It is essential to an accurate balance, that the two arms should be precisely equal in length. The *false balance*, which is sometimes used with a design to defraud, has its arms unequal. The dealer turns such an instrument to his account, both in buying and selling. In buying, he puts his weights on the longer side, for then it takes more than an equivalent to balance them; and, in selling, he puts his weights on the shorter side, because less than an equivalent will produce an equilibrium. The fraud may be detected by making the weights and the merchandize change places. The true weight may be determined from such a balance by the rule given p. 99; or more conveniently, by putting the article whose weight is to be determined into one scale, and counterpoising it with sand, shot, or any convenient substance, in the other scale, and then, removing the article, and finding the exact weight of the counterpoise. It is evident that the weight of the merchandize will be the same as that of the weights employed to balance its counterpoise.

277. The *steelyard* is a lever having unequal arms, in which the same body is made to indicate different weights, by placing it at different distances from the fulcrum. A pair of steelyards has usually two graduated sides for determining smaller or greater weights. It will be seen that on the greater side, the weight is placed nearer the fulcrum. Consequently, the weight indicated by the counterpoise, when at a given distance from the fulcrum, will be proportionally greater. This instrument is very convenient because it requires but one weight. The pressure on the fulcrum, excepting that of the apparatus itself, is only that of the article weighed, whereas in the balance, the fulcrum sustains a double weight. But the balance is susceptible of more sensibility than the steelyard, because the subdivisions of its weights can be effected with a greater degree of precision than the subdivision of the arms of a steelyard.*

* Kater.

278. The *spring steelyard* is a very convenient instrument for weighing, in cases where the subdivisions of the weights are large. It depends on the elasticity of a spiral steel spring, to compress or extend which requires a force proportioned to the degree of compression or extension. The manner of applying it will be easily understood from the representation in fig. 140. After continued use, especially when loaded with heavy weights, the elasticity of the spring is liable to be impaired, and the accuracy of the instrument diminished. When made, however, in the best manner, spring steelyards retain their accuracy for a long time.

Fig. 140.



279. In Fig. 141, is represented a vertical section of a large *Weighing Machine*, such as is used for loads of hay, cotton, or other heavy merchandize.

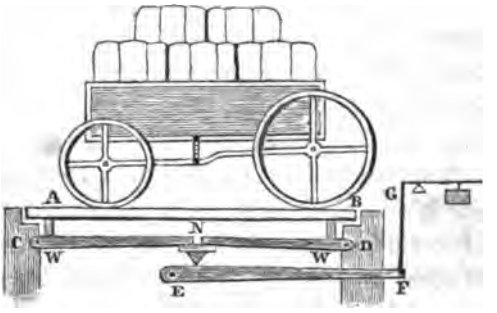
AB, a section of the platform, resting loosely on a frame, H, I.

CN, DN, levers of the second kind, having their fulcrums at C, D, and resting on a bar at N.

W, W, pins which press upward against the platform, when the levers are raised.

EF, a lever likewise of the second kind, having its fulcrum at E, and connected, by a perpendicular arm, with the beam of a pair of steelyards at G.

Fig. 141.



Four levers are usually employed, proceeding from the four corners of an immovable frame, or having their fulcrums firmly set in masonry. The levers all rest on the common support at N.

Suppose a load of merchandize is placed on the platform to be weighed. By the steelyards, we ascertain that the weight exerted at G is 100 pounds, which force is that exerted at F to raise the lever to EF. Supposing, for convenience of computation, that the levers have their longer ten times the length of their shorter arms, then 100 pounds at F balances a force of 1000 pounds at N. This force is still further multiplied by the four levers so as to become 10,000 pounds, which is the weight of the load including that of the platform. If the platform rested on a single lever this would of course sustain a weight of 10,000 pounds; but as the levers severally sustain the same part of the weight, each one bears only one fourth of the load, or 2,500 pounds.

280. When a weight is supported by a lever which rests on two props, the pressure upon both fulcrums is equal to the whole weight. This principle is sometimes applied in ascertaining the weight of a body too heavy for the steelyards. The body is suspended immovably near the center of a pole, and the steelyards are applied to each end of the pole separately, the other end meanwhile resting on its fulcrum. The two weights being added together, make the entire weight of the body. If the body is suspended exactly in the center of the pole, it will be sufficient to obtain the weight of one end and double it. The weight of the lever should, in both cases, be subtracted from the entire weight.

281. Since when a weight is sustained between two props, the *part sustained by each prop is inversely as the distance of the weight from it*, it follows that a load borne on a pole, between two bearers, is distributed in this ratio. As the effort of the bearers, and the direction of the weight are always parallel, it makes no difference whether the pole is parallel to the horizon or inclined to it. Whether the bearers ascend or descend, or move on a level plane, the weight will be shared between them in the same constant ratio. (See Fig. 80.)

282. *Handspikes* and *crowbars* are familiar examples of levers of the first kind. A *hammer* affords an example of the *bent lever;* and shears, pliers, nutcrackers, and all similar instruments, are *double levers;* that is, they consist of two levers united. A pair of shears,

with long handles, like those used by tinners, exhibit very strikingly the increase of power gained by bringing the weight or substance acted on nearer to the fulcrum. The jaws of animals exhibit a similar property. An oar, applied to a boat rowed by hand, a wheel-barrow, and a door shut by the hand applied to the edge remote from the hinges severally, furnish instances of levers of the second kind, where the weight is between the fulcrum and the power.

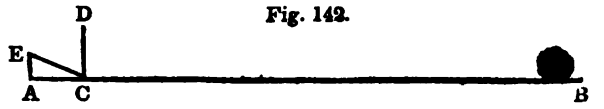
The *crane* is a lever of the second kind, which is much used when great weights are transported for a short distance, as heavy boxes of merchandize from the vessel to the wharf, or great masses of stone from the quarry to a car or boat. An example of the crane, on a small scale, is seen in the apparatus of a kitchen fire-place.

283. When one raises a *ladder* from the ground by one of the lower rounds, the ladder becomes a lever of the third kind, the power being applied between the weight and the prop. Since in all the mechanical powers, the power and weight have equal momenta, and since, in the third kind of lever, the weight has more velocity than the power, the power is as much greater than the weight as the velocity with which it moves is less. The difficulty experienced in raising a ladder from the ground by taking hold of the lowest round, or of shutting a door by applying the hand to the side next to the hinges, shows the mechanical disadvantage under which a lever of this kind acts. Yet it is very useful in cases when it is required to give great *velocity* to the body moved. *Sheep shears*, consist of two levers of this kind united. Here the whole force required is so small that to save it is of no consequence, while so soft and flexible a substance as wool, requires the shears to be moved with considerable velocity. A pair of tongs is composed in the same manner; and therefore it is only a small weight that we can lift with them, especially when the legs are long.

284. One of the most remarkable applications of the third kind of lever, is in the *bones of animals*. These are levers, the joints are the fulcrums, and the muscles are the powers. The muscles are endowed with a strong power of contraction, by which they are made to pull upon a tendon or cord, which is inserted in the bone near the fulcrum. Thus, the fore-arm moves on the joint near the elbow

as a fulcrum, a little below which is inserted a tendon, connected with a muscle between the elbow and the shoulder which gives it motion. The arrangement may be well represented by attaching a small cord to one of the legs of a pair of tongs, near the joint. It will require a considerable force to lift the leg by pulling at the string, especially if the string be pulled in a direction nearly parallel with the leg, as it ought to be, since the tendon which lifts the fore-arm, acts in such a direction with respect to the arm. The muscles therefore act in moving the bones under a double mechanical disadvantage, their force being applied both obliquely and very near the fulcrum. The force which the muscles exert in raising a weight held in the palm of the hand, is enormous, as will be comprehended from the following illustration. Let AB represent the fore-arm moving on the elbow-

Fig. 142.



joint at A, and having the tendon inserted at C, which we will suppose to be one hundred times nearer to A than B is to A. Consequently, a weight of 1 lb. at B, would require a force at C, acting directly upwards of 100 lbs. But the force of the tendon does not act *directly upwards* in the direction of CD, but very obliquely, as in the direction of CE, of which the part EA only can contribute to support the weight. Suppose this part to equal $\frac{1}{10}$th of the whole force CE, and it follows that the muscular force exerted to raise a weight of 1 lb. in the palm of the hand, would, were it to act without any mechanical disadvantage, be sufficient to raise a weight of 1000 lbs. Yet Dr. Young informs us, that a few years ago there was a person at Oxford, who could hold his arm extended for half a minute, with half a hundred weight hanging to his little finger.*

285. But by giving to the muscle the position it has, the greatest possible *compactness of structure* is obtained, while, by making it act so near the fulcrum, what is lost in force, is gained in *velocity;* and while the power acts through a small space, the hands are moved quickly through a great distance. In consequence of the dominion which man can gain over the stronger animals, and especially over

* Young's Lectures on Nat. Phil. 1. 129.

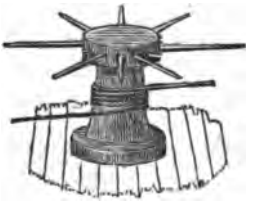
the great powers of Nature, he has little occasion to exert great strength with his naked hands: the celerity of their movements, is to him a far more important endowment.

WHEEL WORK.

286. When a lever is applied to raise a weight, or to overcome a resistance, the space through which it acts at one time is small, and the work must be accomplished by a succession of short and intermitting efforts. The common lever is, therefore, used only in cases where weights are required to be raised through small spaces. When a continuous motion is produced, as in raising ore from the mine, or in weighing the anchor of a vessel, some contrivance must be adopted to remove the intermitting action of the lever, and render it continual. The *wheel and axle*, in its various forms, fully answers this purpose. It may be considered as a revolving lever.

287. In numerous forms of the wheel and axle, the weight is applied by a rope coiled upon the axle; but the manner in which the power is applied is very various, and not often by means of a rope. The circumference of a wheel sometimes carries projecting pins as in Fig. 92, to which the hand is applied to turn the machine. An instance of this occurs in the wheel used in the steerage of a vessel. In the common *windlass* the power is applied by means of a *winch*, which corresponds to the radius of a wheel. (See Fig. 94.) The axis is sometimes placed in a vertical position, and turned by levers

Fig. 143.



moved horizontally. The *capstan* of a ship (Fig. 143.) is an example of this. Levers answering to the radii of a wheel are inserted in holes mortised in the axis, and turned by several men working together. In some cases, as in the *treadmill*, the wheel is turned by the weight of animals walking on the circumference, with a motion like that of ascending a steep hill.

288. The power of the wheel and axle being expressed by the number of times the diameter of the axis is contained in that of the wheel, there are obviously two ways by which this power may be increased; either by increasing the diameter of the wheel, or by diminishing that of the axle. In cases where great power is required, each of these methods is attended with practical inconvenience and difficulty. If the diameter of the wheel is considerably enlarged, the machine will become unwieldy, and the power will work through an unmanageable space. If on the other hand, the power of the machine is increased by reducing the thickness of the axis, the strength of the axis will become insufficient for the support of that weight, the magnitude of which had rendered the increase of the power of the machine necessary. To combine the requisite strength with moderate dimensions and great mechanical power, is therefore impracticable, in the ordinary form of the wheel and axle. This has, however, been accomplished by giving different thicknesses to different parts of the axle, and carrying a rope, which is coiled on the thinner part, through a wheel attached to the weight, and coiling it in the opposite direction on the thicker part, as in Fig. 144. To investigate the proportion of the power to the weight in this case, let Fig. 145 represent a section of the apparatus at right angles to the

Fig. 144.

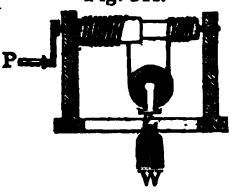
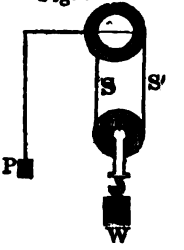


Fig. 145.



axis. The weight is equally suspended by the two parts of the rope S and S′, and therefore each part is stretched by a force equal to half the weight. The momentum of the force which stretches the rope S, is half the weight multiplied by the radius of the thinner part of the axis. This force being on the same side of the center with the power, co-operates with it in supporting the force which stretches S′, and which acts on the other side of the center. Now the momenta of P and S together, must be equal to the momentum of

S′ (Art. 21.); and therefore, if P be multiplied by the radius of the wheel, and added to half the weight multiplied by the radius of the thinner part of the axis, we shall obtain a sum equal to half the weight, multiplied by the radius of the thicker part of the axis. Hence the power multiplied by the radius of the wheel, is equal to half the weight multiplied by the difference of the radii of the thicker and thinner parts of the axis.*

289. A wheel and axle constructed in this manner, is equivalent to an ordinary one, in which the wheel has the same radius, and whose axis has a radius equal to half the difference of the radii of the thicker and thinner parts.† The power of the machine is expressed by the ratio which the radius of the wheel bears to half the difference of these radii; and therefore this power, when the diameter of the wheel is given, does not, as in the ordinary wheel and axle, depend on the smallness of the axis, but on the smallness of the *difference* of the thinner and thicker parts of it. The axle may, therefore, be constructed of such a thickness, as to give it all the requisite strength, and yet the difference of the diameters of its different parts may be so small as to give it all the requisite power.‡

We see here strikingly exemplified the principle, that the weight sustained by a given power may be increased as its velocity is diminished. By inspecting figure 145, it will be seen that the string connected with the thinner part of the axle *unwinds*, while that connected with the thicker part *winds up*, by which means the ascent of the weight may be rendered slow in any degree, and a proportionally greater quantity of matter may be added to balance the constant momentum of the power.

290. It is sometimes desirable to make a *variable* power produce a constant force. This may be done by making its velocity increase

* Let W = weight.
P = power.
R = radius of the wheel.
r = do. of the thicker part.
r' = do. of the thinner part.
Then, $P \times R + \frac{1}{2}W \times r' = \frac{1}{2}W \times r$, $\therefore$
$P \times R = \frac{1}{2}W \times r - \frac{1}{2}W \times r' = \frac{1}{2}W(r - r')$.

† $P \times R = \frac{1}{2}W(r - r') = W \times \frac{1}{2}(r - r')$.

‡ Lardner's El. Mech. p. 181.

as its intensity diminishes. We have an example of this in the reciprocal action between the main-spring and fusee of a watch. (Fig. 146.) The main-spring is coiled up in the box A, and is connected

Fig. 146.



with the fusee B by a chain. When the watch is first wound up, the spring acts with its greatest intensity, but then as the wheel B turns, it uncoils with the least velocity; but on account of the varying diameters of the wheels of the fusee, the velocity is continually increased as the intensity of the spring is diminished. In a similar manner a *varying* weight may be moved by a constant power.

Communication of Motion by Wheel Work.

291. Motion may be transmitted by means of wheel work in several different methods, the principal of which are, the friction of the circumference of one wheel, upon that of another—the friction of a band—and the action of teeth.

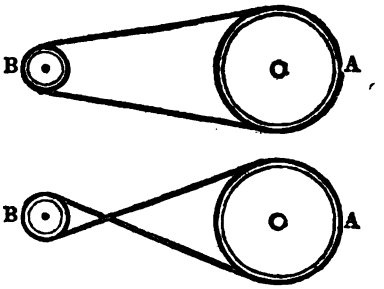
One wheel is sometimes made to turn another, by the mere *friction of the two circumferences.* If the surfaces of both were perfectly smooth, so that all friction was removed, it is obvious that either would slide over the surface of the other, without communicating motion to it. But, on the other hand, if there were any asperities, however small, upon their surfaces, they would become mutually inserted among each other, and neither the wheel nor axle could move without causing the asperities on its edge to encounter those which project from the surface of the other; and thus both wheel and axle would move at the same time. Hence if the surfaces of the wheel and axle are by any means made rough, and pressed together with sufficient force, the motion of either will turn the other, provided the load or resistance be not greater than the force necessary to break off these small projections which produce friction.

292. In some cases where great power is not required, motion is communicated in this way through a train of wheel work, by ren-

dering the surfaces of the wheel and axle rough, either by facing them with buff leather, or with wood cut across the grain. The communication of motion between wheels and axles by friction has the advantage of great smoothness and evenness, and of proceeding with little noise; but this method can be used only in cases where the resistance is not very considerable, and therefore it is seldom adopted in works on a large scale. Dr. Gregory mentions an instance of a saw-mill at Southampton, where the wheels act upon each other, by the contact of the end grain of the wood. The machinery makes very little noise and wears well, having been used not less than twenty years.*

293. Wheel work is extensively moved by the *friction of a band.* When a round cord is used, any degree of friction may be produced, by letting the cord run in a sharp groove at the edge of the wheel. When a strap or flat band is used, its friction may be increased by increasing its width. The surface at the circumference of a wheel

Fig. 147.



which carries a flat band, should not be exactly cylindrical, but a little convex, in which case if the band inclines to slip off at either side, it returns again by the tightening of its inner edge, as may be seen in a turner's lathe. When wheels are connected in the shortest manner by a band, they move in the same direction: if the band be crossed, they will move in opposite directions.† (Fig. 147.) Wheels are sometimes turned by *chains* instead of straps or bands, and are then called *rag wheels.* The chains lay hold upon pins, or

* Gregory's Mech. II. 537.—Lardner's El. Mech. 183.
† Bigelow's Technology, p. 229.

enter into notches, in the circumference of the wheels, so as to cause them to turn simultaneously. They are used when it is necessary that the velocities should be uniform, and where great resistance is to be overcome, as in locomotive steam engines, chain water wheels, &c.*

294. But the most common mode of moving wheel work, is by means of *teeth* cut in the circumference of the wheels. The wheels of necessity turn in opposite directions. The connexion of one toothed wheel with another is called *gearing*. In the formation of teeth, very minute attention must be given to their figure, in order that motion may be communicated from one wheel to another, without rubbing or jarring. If the teeth are ill matched as in figure 148,

Fig. 148.

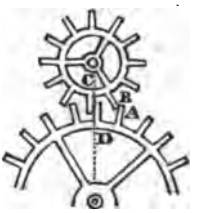
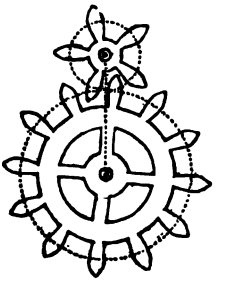


Fig. 149.



when the tooth A comes into contact with B, it acts obliquely upon it, and as it moves, the corner of B slides upon the plane surface of A in such a manner as to produce much friction, and to grind away the side of A, and the end of B. As they approach the position CD, they sustain a jolt the moment their surfaces come into full contact; and after passing the position CD, the same scraping and grinding effect is produced in the opposite direction, until by the revolution of the wheels the teeth become disengaged. To avoid these evils, the surfaces of the teeth are frequently *curved* so as to roll on each other with as little friction, and with as uniform force and velocity as possible. (Fig. 149.) Much pains and skill have been bestowed on this subject by mathematicians, with the view of ascertaining the kinds of curves which fulfil these purposes best.†

* Bigelow's Technology, p. 230.

† See Blake on the form of the teeth of cog wheels, Am. Jour. Science. VIII. 86.

Regulation of Velocity by Wheel Work.

295. Wheel work serves the purpose, not only of forming a convenient communication of motion between the power and the weight, but also of *regulating its velocity*. Thus, when the connection is formed by means of a band, as in figure 147, the velocity of the wheel B, that carries the weight or sustains the pressure may be altered at pleasure, by altering the ratio between the diameters of the two wheels. If the diameters are equal, the wheels will revolve with equal velocity; if A remains the same, while the diameter of B is increased or diminished, the velocity of B will be increased or diminished in the same ratio; or if B remains the same, while the diameter of A is changed, the velocity of B will be changed in the same manner.* We see familiar examples of the application of this principle in the common spinning wheel, and the turner's lathe. In the spinning wheel, a band passes round a large wheel and a small one called a spool, having the spindle for its axis; and in consequence of the great disparity in the size of the wheels, a great velocity is given to the spindle by a comparatively slow revolution of the wheel. In turners' apparatus, machinery for spinning cotton, and the like, a large hollow cylinder or *drum*, is fixed horizontally, which is kept revolving by the moving power, and from which, motion is conveyed by bands to lathes, spindles, &c., to which any required velocity is given, by altering the diameter of the small wheel that is connected with them and turns them. Sometimes a change of velocity is effected by making the drum of a conical shape, and then the velocity imparted to the lathe or the spindle, will be greater or less, according as the band proceeds from the larger or smaller part of the drum.

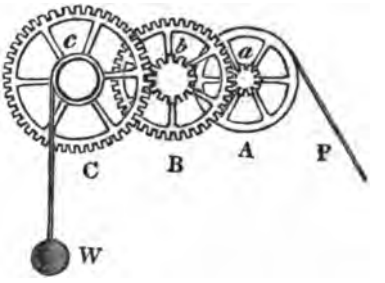
296. A more exact method of regulating the velocity of motion, is by means of *wheels and pinions*.† An example of this kind is

* This would be accurately true, in case the band did not slip or slide; but since it usually does slide more or less, the velocity of the driven wheel is commonly a little less in proportion, than that of the wheel which drives it.—Bigelow, El. Tech. p. 229.

† Pinions are smaller wheels acting on the circumferences of larger. The teeth of a pinion are called *leaves*. They are most commonly raised on the axis of one wheel, and form the communication between that wheel and the next in the series.

seen in Fig. 150, where A, B, C, are three wheels, and *a*, *b*, *c*, a the corresponding pinions. As the leaves of the pinions succes ively pass between the teeth of the wheel, the divisions of the tv circumferences must correspond to each other, and the number

Fig. 150.



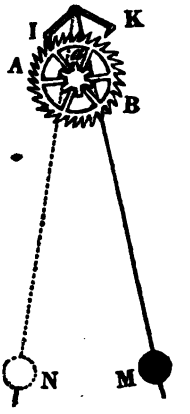
teeth in the wheel will be as much greater than in the pinion, as th circumference of the wheel is greater than that of the pinion. Ther fore it follows, that the number of teeth in a wheel, and of leaves the pinion that acts upon it, expresses the ratio of the circumferen or radius of the wheel to that of the pinion. Hence, in an equili rium, the power multiplied by the product of the numbers expres ing the amount of teeth in all the wheels respectively, is equal to th weight multiplied by the product of the several numbers denotir the leaves in each of the pinions. (Art. 117.)

297. It is farther evident that the velocity of the wheel and that the pinion connected with its circumference, will be inversely as th *number of teeth* in each. Thus in Fig. 150, if the pinion *a* has 1 teeth, and the wheel B has 100, *a* will move ten times as fast as I For the same reason *b* will move ten times as fast as C; so that, this arrangement, the power moves with 100 times the velocity the weight. By varying the ratio between the number of teeth the pinion, and the number of teeth in the wheel with which it connected, we may vary the velocity of any wheel at pleasure.

298. A familiar instance of this is afforded in the mechanism a common clock. A pendulum by falling gains a quantity of m tion sufficient to carry it on the other side to the same height as th from which it fell; and were it not for the resistance of the air ar

the impediments, a pendulum when once set in motion would continue to vibrate by its own inertia, (Art. 19.) and would thus afford, without the aid of any machinery, an exact measure of time. But, in order to continue its vibrations, some small force must be applied to it to compensate for the loss of motion from friction and resistance. This force is applied to the pendulums of clocks by the *weight*, and an analogous force is supplied to the *balance wheel* of watches and chronometers by *springs*. In Fig. 151, let AB be a wheel having 30 teeth, and let N, M, be a pendulum, connected with the wheel by the *pallets* I, K; and to the axis a, let a weight be hung. It is evident that this weight, were there nothing to arrest it, would descend by the force of gravity with accelerated velocity. It *endeavors* thus to descend, and hence exerts the required force on the pallets of the pendulum. For, every time the pendulum performs a double vibration, (returning to the same point from which it set out) a tooth of the wheel escapes,* and the wheel runs down until the next tooth strikes upon the pallet, and thus gives it the impulse which is necessary to keep up the vibrations.

Fig. 151.



299. It would seem therefore that, for beating seconds, only a single wheel is necessary; nor would any more be absolutely indispensable; but in this case the weight would descend so fast, as soon to reach the floor, and the clock would require to be wound up again every few minutes. Hence a series of wheels are interposed between the pendulum and the weight, by which the descent of the latter is retarded upon the principle explained in Art. 297, and the descent of the weight is slower in proportion as the series is more extensive. In cheap clocks, as some of those made with wooden wheels, the series is short, or the number of wheels employed for retarding the descent of the weight is small, and such clocks require frequent winding up; but in clocks of finer workmanship, a greater number of wheels is interposed, and such clocks require to be wound

* Hence this wheel is called the *escapement*.

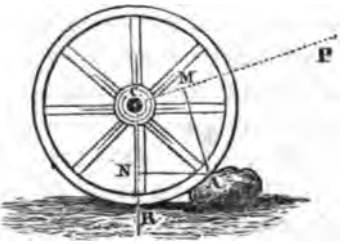
up less frequently. Many go eight days, and some are made to go a whole year without winding.

WHEEL CARRIAGES.

300. When a loaded carriage is moving on a horizontal plane, free of obstacles, the resistance to be overcome does not consist of the weight of the load, directly, but of the friction occasioned by the weight. For, since the weight acts in a direction perpendicular to the plane, it cannot oppose the motion of the carriage in a direction parallel to the plane. Nor would increasing the weight, to any extent, make any difference, were it not that we should thus increase the friction, which (as will be explained more fully hereafter) is proportioned to the weight.

When a carriage wheel is made to *slide* on the ground, (as when a wheel is *locked*,) the whole amount of the friction is encountered without bringing in to our aid any mechanical advantage; but when a wheel turns on its axle, the friction is transferred from the ground to the axle, and each spoke of the wheel successively becomes a lever, turning on the ground as a fulcrum, while the power, or force of the team, is exerted on the end next to the axis. By thus transferring the friction from the ground to the axle, each spoke, in its turn, is made to aid in overcoming that friction. Thus, in Fig. 152,

Fig. 152.



let C be the axis, CP the line of draught, and R the point where the wheel touches the plane. The force applied in the direction CP, acts on CR at C, and turns it on its fulcrum at R. This is the force by which the wheel is made to advance. But the friction on the axle at C, re-acts in the opposite direction, having a leverage equal only to the radius of the axle, while the power which overcomes this, has a leverage equal to the radius of the wheel. Hence,

in the wheel, there is a mechanical advantage gained in overcoming the friction, in the ratio of the radius of the wheel to the radius of the axle. Moreover, the axle may be made of such materials, and lubricated with such substances, as to render the actual amount of friction much less than it would be were the wheel made to slide on the ground.

But wheels have another important advantage, namely, *in overcoming obstacles*; in which case they act on the principle of the bent lever.

Thus let A be an obstacle, as a stone for example. From A let fall the perpendiculars AN, AM, upon CR, CP, and conceive MAN to be a bent lever, turning on A as a fulcrum, the power being applied at M in the direction CP, and the weight resting on N (which supports the center of gravity). Now, the mechanical advantage gained, will be in the ratio of MA to NA. It will therefore be increased (and of course the force necessary to overcome the obstacle be diminished) as the point A is nearer to R; and the mechanical advantage will be lessened as the point A recedes from R. When the obstacle is so large as to make AM only equal AN, then no mechanical advantage is gained, but the whole weight of the load must be lifted by the former; and when AM becomes less than AN, the wheel involves a mechanical disadvantage, and the difficulty of carrying the wheel over the obstacle becomes very great. It is farther obvious that *large wheels* have the mechanical advantage, both as regards overcoming the friction, and overcoming obstacles, in a higher degree than small wheels, since these afford a greater leverage than the others on account of the increased length of the spokes. But in practice very large wheels cannot be employed, since they would be either weak or too heavy, and the increased height of the axle would carry the center of gravity too high, and enhance the danger of upsetting. The difficulty of turning might also render unusually large wheels ineligible; and the axle might be raised so high, as to make the horse draw obliquely downwards and increase the pressure on the ground, whereas the line of draught ought to be so adjusted as to lighten that pressure, especially where the road is soft and yielding.

When a wheel sinks below the surface, the force is rendered strikingly inefficacious from several causes. The fulcrum on which

each spoke successively turns gives way, and diminishes greatly the mechanical advantage otherwise gained by transferring the friction from the ground to the axle, as before explained. Likewise, the mud or sand into which the wheel has sunk, opposes in front of the wheel an obstacle like that represented at A in Fig. 152, while the fulcrum on which the bent lever turns in the effort to lift the wheel over the obstacle gives way as in the other case, and a great part of the mechanical advantage is lost. From these considerations, it is easy to understand the reason of the superior advantages of hard and smooth roads.

301. The *line of draught* should not be horizontal but inclined upwards towards the breast of the horse, in an angle not less than 15 degrees with the horizon. This brings the strain nearly at right angles with the collar, whereas a horizontal draught lifts the collar upwards, by which the force is wasted and the animal is choked.* The angle of draught, however, should be less than the above when the road is very smooth. The general rule is, that the angle should be the same as the inclination of a hill, down which the carriage would roll spontaneously. Consequently, in smooth Macadamized roads, the line of draught should be small, and on railways, nearly horizontal.†

302. The effect of suspending a carriage on *springs*, is to equalize the motion by causing every change to be more gradually communicated to it, and to obviate shocks. Springs are not only useful for the convenience of passengers, but they also diminish the labor of draught; for whenever a wheel strikes a stone, it rises against the pressure of the spring, in many cases without materially disturbing the load, whereas without the spring, the load, or a part of it, must rise with every jolt of the wheel, and will resist the change of place with a degree of inertia proportionate to the weight, and the suddenness of the percussion. Hence springs are highly useful in baggage wagons and other vehicles used for heavy transportation.‡

A pair of horses draw more advantageously abreast than when one is harnessed before the other. In the latter case, the forward horse,

* Fuller on Wheel Carriages.

† Moseley's Mechanics applied to the Arts.

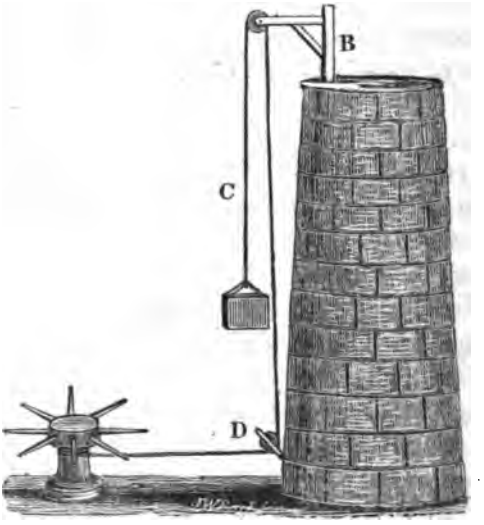
‡ Bigelow, El. Tech. p. 200.

being attached to the ends of the shafts, draws in a line nearly horizontal; consequently he does not act with his whole force upon the load, and moreover expends a part of his force in a vertical pressure on the back of the other horse.

THE PULLEY.

303. Pulleys are divided into FIXED and MOVABLE. In the *fixed* pulley, as has been demonstrated in Art. 120, no mechanical advantage is gained, but its use consists in furnishing a convenient mode of *changing the direction* of the power. Thus, it is far more convenient to raise a bucket from a well by drawing downwards, as is the case when the rope passes over a fixed pulley above the head, than by drawing upwards leaning over the well. By means of the pulley, great facilities are afforded for managing the rigging of a ship. The sails at mast head can be easily raised while the hands stand upon the deck, whereas, without the aid of ropes and pulleys,

Fig. 153.



the same force removed to mast head would operate under very great disadvantages. Similar facilities are afforded by this kind of apparatus for raising heavy weights, as boxes of merchandise, or heavy blocks of stone in building. Fig. 153, represents a convenient

method of elevating large masses of stone in building a high tower, as a lighthouse or monument. The *crane* at B enables the workmen, when the weight is raised, to swing it round to the point where it is to lie, or to a platform near to it. The lower end of the rope CD, is connected with a wheel and axle in the figure, but it is obvious that different methods of applying the power might be adopted, to suit the convenience of the workmen. For example, instead of the wheel and axle we might attach a horse or a yoke of oxen to the rope CD; or we might attach a long *sweep* to the top of the axis, and join a team of horses or cattle to the end of it, and raise the weight by driving them round, as in a common cider mill.

304. *Fire escapes* sometimes consist merely of a pulley fixed near the window of the apartment, around which a rope may be easily placed, having a basket attached to the end. The man seats himself in the basket, grasping, at the same moment, the rope on the other side of the pulley, and thus he lets himself gradually down.

305. The *movable* pulley, by distributing the weight into separate parts, so that it is supported at several different points at once, is attended by a mechanical advantage, proportioned to the number of such points of support. Movable pulleys may be arranged according to several different systems, which increase the efficacy of a given power in different ratios.* It will be observed, however, that the ascent of the weight is in all cases retarded in proportion as the efficacy of a given power is increased. It must be further observed that in using any system of movable pulleys, the whole weight of the pulleys themselves, together with the resistance occasioned by the rigidity and friction of the rope, all act against the power, and so far lessen the weight which it is capable of raising. In the more complex systems of pulleys, it is estimated that at least two thirds of the power is expended on the machinery itself. On account therefore of the slowness of the motion which the weight receives, and the loss of power from the resistance of the ropes and blocks, such systems of pulleys are seldom employed. It is only in raising vast weights, such as large ships, or great masses of stone from a

* See Part I, Arts. 122—125.

quarry, that they are ever used. For managing the rigging of a ship, the combination usually employed consists of not more than two or three movable pulleys. From its portable form, however, its cheapness, and the facility with which it can be applied, especially in changing or modifying the direction of motion, the pulley is one of the most convenient and useful of the mechanical powers.

THE INCLINED PLANE.

306. The inclined plane becomes a mechanical power in consequence of its supporting a part of the weight, and of course leaving only a part to be supported by the power. Thus the power has to encounter only a *portion* of the force of gravity at a time,—a portion which is greater or less, according as the plane is more or less elevated. When a plane is perfectly horizontal, it sustains the entire pressure of a body that rests on it; that is, the pressure on the plane is equal to the whole force of gravity acting on the body. As one end of the plane is elevated, this force is resolved into two, one of which is parallel and the other perpendicular to the plane. In proportion as the plane is more elevated, the part of the force which acts parallel with the plane is increased, until, when the plane becomes perpendicular to the horizon, it no longer sustains any portion of the weight, and the latter descends with the whole force of gravity.

307. The simplest example we have of the application of the Inclined Plane, is that of a plank raised at the hinder end of a cart for the purpose of rolling in heavy articles, as barrels or hogsheads. The force required to roll the body on the plank, setting aside friction, is as much less than that required to lift it perpendicularly, as the height of the plane above the ground is less than its length. Every one knows how much the facility of moving heavy loads is increased by such means, and how the force required to move them is diminished, by increasing the length of the plane while the height remains the same. Long inclined planes, constructed of plank, are frequently employed in building, especially where high walls are built of large masses of stone, the materials being trundled upon the plane on wheel barrows, or transported on heavy rollers. It is even

supposed, that in building the Pyramids of Egypt, the huge masses of stone were elevated on an inclined plane. *Roads* also, except when they are perfectly level, afford examples of this mechanical power. When a horse is drawing a heavy load on a perfectly horizontal plane, what is it that occasions such an expenditure of force? It is not the weight of the load, except so far as that increases the friction; for gravity, acting in a direction perpendicular to the horizon, can oppose no resistance in the direction in which the load is moving. The answer is, that the force of the horse is expended chiefly in overcoming friction, and the resistance of the air. But when a horse is drawing a load up a hill, he has not only these impediments to encounter, but has also to overcome more or less of the force of gravity; that is, he *lifts* such a part of the load as bears to the whole load the same ratio, that the perpendicular height of the hill bears to its length. If the rise is one foot in twenty, he lifts one twentieth of the load, and therefore encounters so much resistance in addition to those which he had to overcome on the horizontal plane. If the ascent were one foot in four, and the load were a ton, the additional force required above what would be necessary on level ground, would be 560 pounds.

308. *Railways* afford another striking exemplification of the principles of the Inclined Plane. By means of them the irregular surface of a country, however hilly and uneven, is reduced to horizontal levels and inclined planes. These are sometimes inclined at so slight an angle, that the tendency of the cars down the plane, is only just sufficient to balance their friction, and they would remain at rest of themselves in any part of the plane, while a small force would move them either way. In other places the Inclined Planes are very steep for a short distance; and the cars ascending them are sometimes drawn up by means of a power (a steam engine, for example,) stationed on the summit, and sometimes cars descending on one side are made to draw up others on the other side, the two being connected by a chain or rope which passes round a pulley on the summit. It is said that on a well constructed horizontal railway, a single horse will draw a load weighing ten tons.*

* Strickland's Reports.

309. The Inclined Plane has been very advantageously substituted for Locks on Canals. The method, in general, is to construct around the falls a railway in the form of an inclined plane; and then the boat being floated into a large cistern of water, the whole is placed on the inclined plane, (the lower end of the cistern being supported so as to keep the surface of the water level,) and is rolled up or down the plane, either by making descending draw up ascending loads, or by drawing up the ascending cistern with its boat by means of machinery. In the latter case, the water-fall itself acting on a wheel, may be made to afford the requisite power.*

THE SCREW.

310. When a road, instead of ascending a hill directly, winds round it to the summit, so as to lengthen the inclined plane, and thus aid the moving force, the Inclined Plane becomes a Screw. In the same manner a pair of stairs, winding around the sides of a cylindrical tower, either within or without, affords an instance of an inclined plane so modified as to become a screw. These examples show the strong analogy which subsists between these two mechanical powers; or rather, they show that the screw is a mere modification of the Inclined Plane.

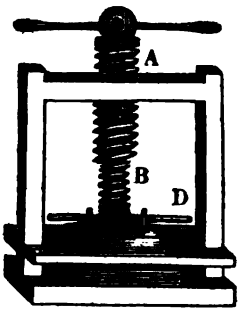
311. The Screw is generally employed when severe pressure is to be exerted through small spaces, and is therefore the agent in most presses. Being subject to great loss from friction, (upon which however its chief utility depends, as will be shown hereafter,) it usually exerts but a small power of itself, but derives its principal efficacy from the lever, or from wheel work, with which it is very easily combined. Thus, in Fig. 154, were the power applied directly to the screw, the mechanical advantage gained would hardly more than compensate for the loss by friction; but by means of the lever, (which may be lengthened or shortened at pleasure,) the power is greatly increased. Also, by means of the *endless screw*, Fig. 99, combined with the wheel and axle, a very powerful force may be exerted; and as the mechanical power of the screw depends upon the relative

* See "Report on Renwick's Inclined Plane," *Journal of the Franklin Institute, Vol. II.*

magnitude of the circumference through which the power revolves, and the distance between the threads, (Art. 139.) it is evident that, to increase the efficacy of the machine, we must either increase the length of the lever by which the power acts, or diminish the distance between the threads. Although, in theory, there is no limit to the increase of the mechanical efficacy by these means, yet practical inconvenience arises from the great space over which a very long lever traverses. If, on the other hand, the power of the machine be increased by diminishing the distance between the threads, and of course their size, the thread will become too slender to bear a great resistance. The cases in which it is necessary to increase the power of the machine, being those in which the greatest resistances are to be overcome, the object will evidently be defeated, if the means chosen to increase that power, deprives the machine of the strength which is necessary to sustain the force to which it is to be submitted.*

312. These inconveniences are remedied by *Hunter's Screw*, which, while it gives to the machine all the requisite strength and compactness, allows it to have an almost unlimited degree of mechanical efficacy. This contrivance, which is represented in figure 154, acts upon a principle similar to that of the wheel and axle, as

Fig. 154.



represented in figure 144, where the efficacy of the power is increased by diminishing the velocity of the weight, which is accomplished by making the rope unwind on one side while it winds up, with somewhat greater speed, on the other side. In the case before us, the screw is likewise composed of a smaller and a larger thread,

* Lardner's El. Mech. p. 220.

tion. All cutting and piercing instruments, such as knives, razors, scissors, chisels, nails, pins, needles, awls, &c. are wedges. The angle of the wedge in these cases, is more or less acute, according to the purpose to which it is applied. In determining this, two things are to be considered—the mechanical power, which is increased by diminishing the angle of the wedge, (Art. 146.); and the strength of the tool, which is always diminished by the same cause. There is, therefore, a practical limit to the increase of the power, and that degree of sharpness only is to be given to the tool, which is consistent with the strength requisite for the purpose to which it is to be applied. In tools intended for cutting wood, the angle is generally about 30°; for iron it is from 50° to 60°; and for brass, from 80° to 90°. Tools which act by pressure may be made more acute than those which are driven by a blow; and, in general, the softer and more yielding the substance to be divided is, and the less the power required to act upon it, the more acute the wedge may be constructed.*

316. In many cases, the utility of the wedge depends on that which is entirely omitted in the theory, viz. the *friction* which arises between its surface and the substance which it divides. This is the case when pins, bolts, or nails, are used for binding the parts of structures together; in which case were it not for the friction, they would recoil from their places, and fail to produce the desired effect. Even when the wedge is used as a mechanical engine, the presence of friction is absolutely indispensable to its practical utility. The power generally acts by successive blows, and is therefore subject to constant intermission, and but for the friction, the wedge would recoil between the intervals of the blows with as much force as it had been driven forward, and the object of the labor would be continually frustrated.

GENERAL REMARKS ON MACHINERY.

317. Archimedes is said to have boasted to King Hiero, that "if he would give him a place to fix his machine, (a *που στῶ*,) he would

* Lardner.

move the world." Yet there can be no machine by the aid of which Archimedes could move the world, in any other way, than by moving, himself, over as much more space than that over which he moved the earth, as his weight was less than that of the whole earth. If Archimedes had received the place he desired, and had also employed what was equally desirable, a machine which operated free of all resistance, he must have moved with the velocity of a cannon ball, to have shifted the earth only the 27 millionth part of an inch in a million of years.*

318. Machines are divided into two classes, those intended simply to *sustain* a weight, and those intended to *move* it. In machines of the first class, estimating the effect by the weight sustained, it is evident that the efficacy of the power is increased. By means of a lever, for example, a man may sustain a weight ten times as great as he could by his unaided strength. We may perceive, however, on closer examination, that he does not in fact bear the whole weight, but only one tenth part of it. Let it be a lever of the second kind, where the weight is ten times nearer the fulcrum than the end is to which the power is applied. Now, the hand that supports this end performs the same office as the second fulcrum in a lever of the first kind; and since (Art. 102.) the pressure on each fulcrum is inversely as its distance from the weight, therefore, in the present case, nine parts out of the ten are borne by the prop, and only one by the power. If (says Carnot,) Archimedes had obtained his "fixed point," it would not, in reality, have been Archimedes, but the fixed point that would have sustained the earth.

In machines of the second class, the effect is not estimated simply by the weight moved, but it requires to be taken into the account the time occupied in moving it a certain distance, that is, the *velocity*. Hence, the effect of moving powers is estimated by the product of the weight moved multiplied by the velocity, or it is measured by the *momentum* produced. Moreover, in the former case, all resistance from friction, the rigidity of ropes, and so on, conspire with the power in sustaining the weight; but in machines of the second class, all such resistances oppose the action of the

* Ed. Encyc. XII. 578.

power, and require a greater power for a given weight, than would be necessary, if the power were applied directly to the weight.*

319. Hence it will be inferred, that no *momentum*, or effective force, is gained by any of the mechanical powers, or by any machine. If a man, with his naked hands, can lift to a given height, as one foot, only 150 pounds in one second, it is impossible for him to perform any more labor than this by any mechanical contrivances.† On the contrary, when the structure of the machine is complicated, there is a loss of force, by employing the machine instead of the naked hands, proportioned to the resistance of the parts of the machine itself. It is to be remarked, however, that this doctrine proceeds on the supposition that the *useful effect* produced is estimated from the joint product of the *force*, *velocity*, and *time*. Thus, $F \times T \times V = \frac{1}{2}F \times 2T \times V = 2F \times \frac{1}{2}T \times V = F \times \frac{1}{2}T \times 2V$, &c.‡ A convenient method of estimating different forces is to draw a heavy weight out of a well, by a rope passing horizontally over a fixed pulley, near the top of the well. Suppose that a man can draw up a rock weighing 100 pounds, through the space of 50 feet in one minute. He would, of course, be able to draw up ten such masses in ten minutes, weighing in all 1000 pounds. Now by passing the rope over five pulleys, (allowing nothing for the friction of the pulleys,) he might with the same force lift the whole 1000 pounds at once; but it would rise ten times as slowly as the 100 pounds did before, and consequently would be ten minutes in reaching the top. Therefore, in a given time, it appears that the man would raise the same weight through a given space, with or without the aid of machinery. In the former case, the 100 pounds might have been raised during the ten minutes through the space of 500 instead of 50 feet; but $100 \times 500 \times 10 = 1000 \times 50 \times 10$: so that the labor performed would have been the same in both cases. Let us suppose that P is a power amounting to an ounce, and that W is a weight amounting to 50 ounces, and that P elevates W by means of a machine. In virtue of the property already stated, it follows, that while P moves through 50 feet, W will be moved through 1

* Venturoli's Mechanics, p. 164.

† Emerson's Mechanics, p. 150. Cavallo, Nat. Phil. I. 251.

‡ Gregory, I. 343.

foot; but in moving P through 50 feet, *fifty distinct efforts* are made, by each of which, if applied directly, 1 ounce would be moved through 1 foot.*

320. *What then*, it may be asked, *are the advantages gained by Machinery?* The advantages still are very great, for the following reasons.

(1.) By the aid of machinery *we can frequently apply our force to much better purpose.* Thus in lifting a weight out of a well, or in raising ore out of a mine, it is obvious with how much more effect a man can work at the arm of a windlass, than he could draw directly upon the rope by stooping over the well. So in raising a rock from its bed by means of a handspike or crowbar, we can easily see how much more effectually we can bring our force to bear upon it than we could do with our naked hands.

(2.) By the aid of machinery, a man may be able to perform works to which his naked strength would be wholly incompetent. Thus, as in the preceding example, one might be able to lift a rock from its bed with a handspike, upon which he could make no impression with his naked hands: or, by means of pulleys, he might raise a box of merchandize from the hold of a ship, which he could not start at all with his unassisted force. In each of these cases, *if the weight could be divided into small parcels,* and if the force could be as advantageously applied without machinery as with it, the labor would be performed as easily in a given time in one way as in the other. But it might not be possible or at least convenient thus to divide it. Or if, instead of dividing it into a number of parcels, the *same number of men* could act directly upon the weight at once, the amount of labor which they would all exert in raising the weight without machinery, would be the same as that which the single man before supposed would exert with his machinery. But it might not be convenient to assemble so many hands at a time; or perhaps such a number could not work advantageously together. A farmer has many occasions for lifting or removing great weights when his laborers are not more in number than two or three in all. These must therefore perform the labor of 50 times as many men by being 50

* Lardner's Mechanics, p. 162.

times as long about it. Thus, in the example given on page 130, of a combination of the mechanical powers employed to haul a ship on the stocks, where a single man turning on a winch, with the force of 100 pounds exerts a force on the ship amounting to 161½ tons, the ship would move as much slower than the hand as 100 pounds is less than 161½ tons; and consequently a great length of time would be required for an individual to perform this labor, even supposing no resistance were encountered from the machinery itself.

(3.) Machinery frequently enables a man to exert his *whole force* in circumstances where, without such aid, he could employ but a part of it. Thus, in winding silk or thread, to turn a single reel might not require one fiftieth part of the force which the laborer is capable of exerting. Suitable machinery would enable him to turn fifty spools at once.

(4.) But the most striking advantage of Machinery, is not found in the facilities which it lends to the personal strength of man. It lies in this, that it affords the means of calling in to his assistance the superior powers of the horse and the ox, of water, of wind, and especially of steam. Here we find the excellence of mechanical contrivances fully exhibited; and no where else has the inventive genius of man displayed itself to so great advantage. But here, as in all other cases, the various combinations of mechanical powers *produce* no force: they only *apply* it. They form the communication between the moving power and the body moved; and while the power itself may be incapable of acting except in one direction, we are able by means of cranks, levers, and toothed wheels, to direct and modify that force to suit our convenience or necessities. Every one may see examples of this in the construction of the most common saw mill or flour mill, turned by water. In a mill for grinding wheat, the stones are required to move horizontally, while the action of the water-fall is perpendicular. We therefore receive the whole force on the circumference of a wheel, and transmit it through several intermediate wheels to the revolving stone, where the grinding is performed. So in a saw mill, the water first communicates a *rotary* motion to the wheel, and this motion is converted by means of a crank into what is called a *reciprocating* motion, as that of the saw in its ascent and descent. By means of wheel work the *velocity* of the moving body is increased or diminished at pleasure.

321. In short, machines enable us to form a convenient communication between the power and the weight; to give to the weight any required direction or velocity; to apply force to the best advantage; to vary the circumstances of velocity and time as the amount of our force may require; and to bring to our aid the great moving powers that exist in nature. Our next object, therefore, will be to see by what particular methods, these several purposes are accomplished.

CHAPTER IV.

REGULATION OF MACHINERY, AND CONTRIVANCES FOR MODIFYING MOTION.

322. It is highly important to the successful operation of any machine, that its motion should be regular and uniform. Jolts and irregular movements, waste the power, wear upon the machine, and perform the work unevenly. The sources of irregularity are various, but they are chiefly the three following, viz. variations in the power, variations in the weight or resistance, and changes of velocity in parts of the machine itself. Thus in the steam engine, the fire may burn with more or less intensity and produce corresponding quantities of the moving power; the load to be carried (as that of a steam boat,) may be much greater at one time than at another, and be subject to sudden changes; and the motion of the piston, which carries the machinery, ceases altogether at the highest and lowest points, and would move a machine by *hitches* or separate impulses, were there no contrivance connected with it for keeping up a uniform motion.

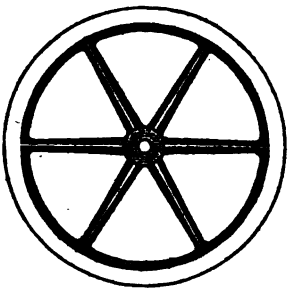
323. The kinds of apparatus employed to obviate these difficulties, and to secure uniform movements to machines, are, in general, called REGULATORS. Large machines or engines themselves, in consequence of their inertia, acquire and maintain, to a considerable extent, uniformity of motion. A flour mill carried by water, when it has acquired a certain rate of going, will not suddenly change that rate by any alteration in the force of the steam; and a ship sailing

between the opposite forces, arising from the impulse of the wind and the resistance of the water, will move steadily along, notwithstanding the breeze that carries it may fluctuate continually. We can see this principle sometimes operating on a smaller scale. A grindstone turned by a winch moves steadily, although the force applied at one part of the revolution is much greater than at another. Large grindstones exhibit the advantage of this principle much more than small ones. But in many instances, this natural tendency towards uniform motion is not sufficient, and artificial contrivances are introduced expressly for this purpose. As examples of Regulators we may especially notice three, the Pendulum, the Fly Wheel, and the Governor.

324. The *Pendulum*, by its equal vibrations, communicates to delicate machinery a motion extremely regular, and hence its application to the measurement of time.

The *Fly Wheel* affords the most common and effectual method of equalizing motion, especially in heavy kinds of machinery. It consists of a heavy wheel (Fig. 155.) affording as much weight as possible under as small a surface, in order that the inertia may be great, while the resistance from the air is small. It is therefore usually a heavy hoop of iron with thick bars of the same metal. The Fly is balanced on its axis, and so connected with the machinery as

Fig. 155.



to turn rapidly around with it, and receiving a constant impulse from the moving power, it becomes a magazine or repository of motion. Consequently, by its inertia, it is ready to supply any deficiency of power that may arise from the sudden diminution of the moving force, or to check any sudden impulse which may result from an accidental excess of that force. Suppose, for example, the handle

of a pump to be connected with a water wheel, and to be carried by it. Here the power, namely the water-fall, is constant, while the weight is subject to continual alternations, amounting to a heavy load as the piston is ascending, but opposing scarcely any resistance while the piston is descending. The motion, therefore, would vary between nothing and a highly accelerated velocity, and the machinery would be subject to constant strains and jolts. A Fly prevents these alternations and renders the ascent and descent of the piston nearly uniform. In pile engines or stamping mills, a team of horses is sometimes employed to raise a heavy weight, which, when at a certain elevation, is suddenly disengaged and falls with great force. As the disengagement is instantaneous, the horses would instantly tumble down were not their motion checked by some contrivance which should prevent the machinery from receiving any sudden increase of velocity. This purpose is completely answered by the Fly.*

325. Beside the use of the Fly Wheel in regulating the action of Machinery, it is employed for the purpose of *accumulating* successive exertions of a power so as to produce a much more forcible effect by their aggregation than could possibly be done by their separate actions. If a small force be repeatedly applied in giving rotation to a Fly Wheel, and be continued until the wheel has acquired a very considerable velocity, such a quantity of force will be at length accumulated in its circumference, as to overcome resistance and produce effects utterly disproportionate to the immediate action of the original force. Thus it would be very easy in a few seconds, by the mere action of a man's arm, to impart to the circumference of a Fly Wheel, a force which would give an impulse to a musket ball equal to that which it receives from a full charge of powder.†

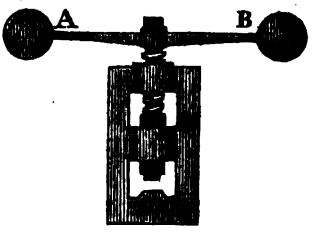
326. The same principle explains the force with which a stone may be projected from a sling. The thong is swung several times around by the arm until a considerable portion of force is accumulated, and then it is projected with all the collected force. If a heavy leaden ball be attached to the end of a strong piece of cane or whalebone, it may easily be driven through a board: by taking the end of the rod remote from the ball in the hand, and striking the

* Gregory's Mech. II. 14. † Library of Useful Knowledge.

board a smart blow with the end bearing the ball, such a velocity may easily be given to the ball as will drive it through the board.*

327. The astonishing effects of a Fly Wheel, as an accumulator of force, have led some into the error of supposing that such an apparatus *increases the actual force of a machine.* So far from this, since a Fly cannot act without friction and resistance from the air, a portion of the actual moving force must unavoidably be lost by the use of this appendage. In cases, however, where a Fly is properly adjusted and applied, this loss of power is inconsiderable, compared with the advantageous distribution of what remains. As an accumulator of force, a Fly can never have more force than has been applied to put it in motion. In this respect it is analogous to an elastic spring. In bending a spring, a gradual expenditure of power is necessary. On the recoil, this power is exerted in a much shorter time than that consumed in its production, but its total amount is not altered. In this way the Fly Wheel is used. Thus, in mills for rolling metal, the water wheel or other moving power, is allowed for some time, to act upon the Fly alone, no load being placed upon the machine. A force is thus gained which is sufficient to roll a large piece of metal, to which, without such means, the mill would be quite inadequate. In the same manner, a force may be gained by the arm of a man acting on a Fly for a few seconds, sufficient to impress an image on a piece of metal by an instantaneous stroke.

Fig. 156.



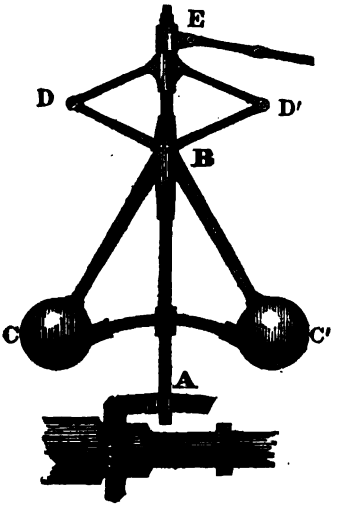
The Fly is, therefore, the principal agent in coining presses. Its power is often transmitted to the working point by means of a screw. At the extremities of the cross arm AB, (Fig. 156.) which works the screw, two heavy balls of metal are placed. When the arm AB is whirled round, those masses of metal acquire a momentum, by which the screw, being driven forward, urges the die with im-

* Library of Useful Knowledge.

mense force against the substance destined to receive the impression. Some engines used in coining have Flies with arms four feet long, bearing one hundred weight at each of their extremities. By turning such an arm at the rate of one entire circumference in a second, the die will be driven against the metal with the same force as that with which 7500 pounds weight would fall from the height of 16 feet; an enormous power, if the simplicity and compactness of the machine be considered. By the action of a Fly, working in this manner, is produced the open work of fenders, fire grates, and sometimes ornamental articles wrought in metal. The cutting tool, shaped according to the pattern to be executed, is attached to the end of a screw; and the metal being held in a proper position beneath it, the Fly is made to urge the tool downwards with such force as to stamp out pieces of the required figure. When the pattern is complicated, and it is necessary to preserve with exactness the relative situation of its different parts, a number of punches are impelled together, so as to strike the entire piece of metal at the same instant, and in this manner the most elaborate work is executed by a single stroke of the hand.*

328. To maintain a uniform velocity with a varying resistance, one of the most beautiful contrivances ever used is the *Governor*, an

Fig. 157.



* Lardner.

instrument used in mill work, but the application of which is most conspicuous in the steam engine, when that machine is applied to manufacturing purposes. The principle on which the efficacy of this instrument depends, is easily explained. Let AB be a vertical axis which is made to revolve by the wheel A, acted on by the other parts of the machinery, and so that it always revolves with a velocity proportional to that of the fly wheel. Two heavy balls C, C′ are attached to metal rods, which work on a pivot at B, so that they are capable of receding from the axis AB. As they recede from the axis, the joints D, D′ recede from one another, and the joint E is drawn down. This joint E is connected with the end of a lever or a system of levers, the action of which we shall presently explain.

329. Now by the revolution of the spindle or axis AB, the balls C, C′ acquire an obvious tendency to fly off from the axis, and this tendency is resisted by their weight; so that, when the instrument is revolving with a certain velocity, the balls remain suspended, and neither move to or from the axis. A greater velocity, by giving a greater centrifugal force, would cause the balls to fly farther off, (Art. 227.) and a less velocity would cause them to fall towards the axis. This is strictly true only when the range of the balls is small, compared with the length of the rods to which they are attached, which, however, is always the case in practice. If therefore, the action of the levers with which the joint E is connected is directed upon the first mover in such a manner, that its energy is diminished when E is depressed, and increased when E is elevated, it is plain that the uniformity of velocity which is sought may be obtained. Let us suppose that the levers on which E works communicate with a valve which admits steam to the piston of a steam engine, to which this Governor is applied; and suppose that when E is raised, and the balls C, C′ rest in their seats, the valve is fully open, so as to allow the steam to flow in a full stream to the piston; but that, according as E is depressed, the levers gradually close the valve, so as to admit the steam in a constantly diminished quantity. Now suppose that the engine has been working twenty printing presses, and that the action of ten of them is suddenly suspended. The engine thus loses half its load, and would if the same power of

steam continued to be admitted, move with about twice its former velocity. But the moment an increased velocity is perceived in the machine, the balls C, C′ recede from the axis, draw down the joint E, partially close the valve, and check the supply of steam to the cylinder. The impelling power is thus diminished; and if it be diminished in exactly the same degree as the load, the machine will move with its former velocity; but if it should, at first, be more diminished, the velocity will be less than its former velocity, and the balls will again move towards the axis and open the valve, and will, at length, settle into that position in which the steam admitted to the cylinder is exactly proportioned to the load on the machine; and the proper velocity will thus be restored.*

CONTRIVANCES FOR MODIFYING MOTION.

330. In Chapter III. we have already explained the mode in which motion is communicated, and its velocity regulated, by *wheel work*. We proceed now to consider a few examples of the more special contrivances by which motion is modified to suit particular purposes, recommending it to the student of mechanics to make himself acquainted with other contrivances of the same nature, by the actual inspection of machinery as opportunity may offer.

331. The motion required for a particular purpose may be *rectilinear*, as that of a carriage or bucket drawn out of a well, or *rotary* as in ordinary wheel work, or *reciprocating* as in a saw mill, or a pendulum.

The simplest mode of producing rectilinear motion is by means of a rope or chain, instances of which are familiar to every one. The simplest mode of *changing the direction* is by means of pulleys; but toothed wheels are also extensively employed for the same purpose.* The connexion of one toothed wheel with another is called *gearing*. (Art. 294.) When both wheels with their teeth are in the direction of the same plane, it is called *spur gearing*, (Fig. 148, 49, and 50.); if the teeth, instead of being cut on the circumference in a direction parallel to the axis, are cut obliquely, so that if continued they would pass round the axis like a screw, it is called *spiral gearing*, (Fig.

* Library of Useful Knowledge. * Emerson's Mechanics, Prop. CX.

158.); and when wheels are not situated in the same or parallel planes, but form an angle with each other, the wheels themselves are sometimes shaped like frusta of cones, having their teeth cut obliquely, and converging towards the point where the apex of the cone would be situated, and it is then called *bevel gearing*,* (Fig. 159.)

Fig. 158.

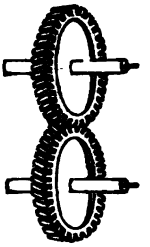


Fig. 159.

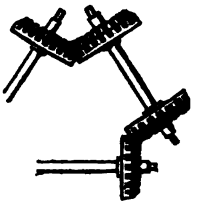
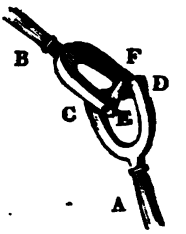


Fig. 160.



332. The *universal joint* consists of two shafts or arms, each terminating in a semicircle, and connected together by means of a cross upon which each semicircle is hinged. (Fig. 160.) When one shaft is turned, either to the right or left, the other shaft turns in the same direction.

The *ratchet wheel*, (Fig. 161.) is used to prevent motion in one direction while it permits it in the opposite. The teeth are cut with their faces inclining as in the figure, and a *catch* is so placed as to stop the wheel in one direction, while it slides over the teeth without obstruction in the opposite direction.

Fig. 161.

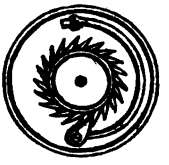
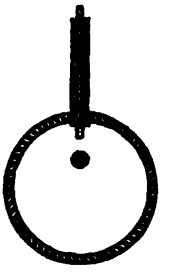


Fig. 162.

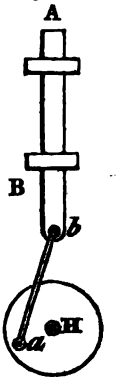


333. The *eccentric wheel* (Fig. 162.) revolves about an axis which is more or less removed from the center, and, consequently, the different portions of the circumference move with different de-

* Bigelow's Tech. 232.

grees of velocity. Hence if this wheel is made to act upon a shaft or pinion, as in the figure, it will carry it with a corresponding movement. In orreries, such wheels are employed for indicating the variable velocities of the heavenly bodies, as they revolve about their centers of motion.

Fig. 163.



334. RECIPROCATING MOTION is produced in various ways. The most common method is by means of the *crank*. In figure 163, a shaft AB is urged backwards or forwards, (either vertically or horizontally,) by means of the crank *ab*, moving on a wheel H, which may be turned by water or any other power acting at H. By considering the different positions of the crank during the revolution of the wheel, it will be readily seen that the shaft will move up and down like the saw in a saw mill, or backwards and forwards, a use to which it is applied in polishing plane surfaces, as marble.

The motion produced by cranks is easy and gradual, being most rapid in the middle of the stroke, and gradually retarded towards the extremes; so that shocks and jolts in the moving machinery are diminished, or wholly prevented by their use.*

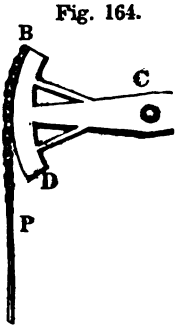
The steam engine, as seen in steamboats, furnishes to the student of Mechanics a valuable opportunity of observing various contrivances for producing, regulating, and modifying motion. Levers and wheels of various kinds and variously connected; fly wheels and cranks; circular and reciprocating motions; and numerous other particulars which appertain to the "elements of machinery," are there seen to the greatest advantage.

335. The *arch head* (Fig. 164.) is a circular form given to the end of a lever, that moves on a gudgeon or pivot in a vertical plane, (like the working beam of a steam engine,) by means of which the piston, or weight, whatever it be, is made to ascend and descend perpendicularly in a *straight* line, with a *uniform* motion, while the end of the lever itself works in the arc of a circle, and while the power, if fixed in the usual way at one point at the extremity of the

* Bigelow.

lever, would lose a part of its efficacy, as its direction became oblique to that of the lever. In the figure, BD represents the arch head of a working beam, which turns on a gudgeon at C. At the top B is applied a flexible chain resting loosely upon the arch, and

Fig. 164.

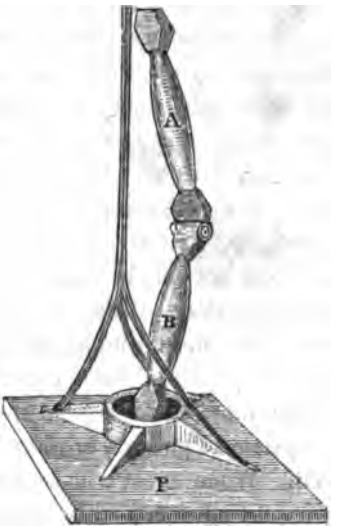


always maintaining a *vertical position* in every situation of the beam; and, being a tangent to the circle at the point of contact, always acting at right angles to the lever, and consequently imparting to the weight a *uniform* motion. In cases where force is to be exerted only in one direction, as in working a pump, the power is applied while the piston is descending and drags up the other end of the lever. Here, in the descent of the working end, so small a resistance is encountered, that it is sufficient to give a slight preponderance to that end, and it will drag up the piston after the moving force is withdrawn, which carried it down. But in cases where a continued force is required, both in the ascent and descent of the piston, (as in most kinds of manufacturing operations,) then the flexible chain, being incapable of forcing the weight upwards, is inapplicable, but the object is attained by arming the arch head with *teeth* that act on *rack work* in the upper end of the piston rod.

336. The *knee joint* (Fig. 165.) is a contrivance by which a constantly increasing resistance, is overcome by a force which acts nearly uniformly. In the figure, we have a representation of it as exemplified in printing presses. Here, when the *platen* (P) first descends upon the sheet, the resistance is very slight, but as the lever is pulled still further, the resistance rapidly increases, until, without mechanical aid, it would require a laborious and exhausting effort, to render the contact sufficiently close to give a perfect im-

pression. This difficulty is completely obviated, by means of the combination of levers resembling the knee joint, where the efficacy of the power is continually increased as the levers approach nearer to the same straight line; so that without any additional effort on

Fig. 165.



the part of the workman, the pressure is augmented a thousand fold. The action of this joint is exemplified in opening a pair of compasses, or a Gunter's scale which opens on a joint at the center. At first, the *thrust* exerted by the ends is slight; but as the two parts approach the direction of a straight line with each other, the thrust rapidly increases, until it becomes immensely great.*

We shall conclude this chapter with a few practical rules for the construction and management of machinery, selected chiefly from Emerson's Mechanics.

337. If the given power is not able to overcome the given resistance when *directly* applied, then we must employ a longer time by making the power move over a greater space than the weight, in order to make the momentum of the power exceed that of the

* See an able article on this subject by Prof. Fisher, *Amer. Journal of Science, Vol. III. p.* 311.

weight. This is effected by means of machinery. We must consider that the power must have a momentum which is equal to the amount of all the resistances to be overcome, in order just to *sustain* or counterbalance the weight; and to put the machine *in motion*, a greater power must be allowed, more or less, according to the velocity with which the machine is required to move. The ratio between the power and the weight in a given machine, when in equilibrium, may be ascertained by observing the *comparative spaces* over which they move in a given time; the power being as much less than the weight, as its space is greater. A due proportion between the two must also be observed; for if the machine or engine is competent to overcome the resistance, and perform its work in a convenient time, it is sufficient for the end proposed; and to increase the power any further, must not only be a needless expense, but the engine would lose time in working. If a weight is to be moved but a little way, the lever is the most simple, easy, and ready instrument. If the weight be very great, the common screw is preferable. But if the weight is to be moved over a great space, the wheel and axle, the pulley, or a system of pulleys, or the endless screw, (Fig. 99.) is to be employed. When great wheels are wrought by men or cattle, their axes are most advantageously placed perpendicularly, as in figure 153; if wrought by water, they are placed horizontally.

338. But most machines are combinations of some or all of the mechanical powers. Thus the lever is combined with the screw in a common press; the wheel and axle with pulleys in various ways, and with the endless screw; pulleys are combined with pulleys, and wheels with wheels. The wedge is the only one among the mechanical powers, that does not admit of combination with others. In wheels with teeth, the number of teeth that play together in two wheels ought to be *prime* to each other, that the same teeth may not meet at every revolution, but as seldom as possible. The strength of every part of a machine ought to be made proportional to the stress it is to bear; and no part must be stronger or heavier than is necessary, for all superfluous matter is nothing but a dead weight upon the machine, and serves for nothing but to clog its motion. The accomplished mechanic contrives all the parts to last equally well, so that when the machine fails, every part shall be worn out.

339. Every machine ought to be made of as *few parts*, and as simple as possible, to answer its purpose; not only because the expense of making and repairing will be less, but it will be less liable to get out of order. Any useless motions also waste some portion of the power. Uniformity or steadiness of motion is carefully to be preserved. All these advantages are more easily attained in large than in small machines. All mechanical errors have a less ratio to the motion of the machine in great machines than in small ones, and these will therefore work with more uniformity and exactness, although being proportionally weaker, they are less able to resist any violent shocks.*

CHAPTER V.

OF FRICTION.

340. The term Friction, in its usual acceptation, being generally understood, we have already employed it in the foregoing pages, but we proceed now to inquire more particularly respecting its nature, the laws of its action, and its effects upon machines.

In investigating the mathematical principles of Mechanics, we first proceed on the supposition that the forces in question act without any impediments; that the surfaces which move in contact are perfectly polished and suffer no friction; that axes and pivots are mathematical lines and points; that ropes are perfectly flexible; and, in short, that the power is transmitted through the machine to the working point without sustaining the least loss or diminution. Great simplicity is attained by first bringing the subject to this ideal standard of perfection, and afterwards making suitable allowances for all those causes which operate in any given case to prevent the perfect action of a machine.

341. Surfaces meet with a certain degree of resistance in moving on each other, in consequence of *the mutual cohesion of the parts*,

* Emerson's Mechanics, 4to. p. 175.

(2.) Friction is proportioned to the *pressure*. If the pressure of the brick be doubled or trebled by laying weights upon it, the amount of friction will be increased in the same ratio.

(3.) Friction is increased by bodies *remaining for some time in contact with each other*. In some cases it does not reach its maximum under four or five days. This principle therefore affects slow motions much more than such as are rapid. In the mutual contact of metals, the friction attains its maximum almost instantaneously. But when metal rubs against wood, or one piece of wood against another, the friction is always increased by resting. Two pieces of wood acquire the utmost friction in an hour or two; while iron running on oak will have its friction augmenting for five or six days. The application of a coat of tallow seems to protract the limit of friction. This limit is attained by the greased surfaces of iron and copper in four minutes; while pieces of wood, treated in the same way, will have their friction gradually augmented during nine or ten days.*

(4.) The friction is less between surfaces of *different* kinds of matter, than between those of the *same* kind. Copper slides on copper, or brass on brass, with greater difficulty than copper on brass; and it is a general rule never to let two substances of the same hardness move upon each other. To this rule cast steel is said to form the only exception; in other cases, pivots revolve with less resistance on either harder or softer substances, than upon those of the same material with themselves.† When between the surfaces of wood newly planed, the friction would be equal to one half the pressure, and when between two metallic surfaces it would be equal to one fourth, between the wood and metal it would amount to only one fifth the pressure.‡

(5.) Friction is much greater at the first moving of a load, than after it is brought freely into motion. In many instances, it is reduced, when a body has attained its final velocity, to less than one half of what it was first.§ With regard to different degrees of velocity in moving bodies, it is a *general* principle, that *the friction is the same for all velocities*; that a carriage, for example, in travelling

* Leslie, El. Nat. Phil. I. 225.
† Allen's Mechanics, p. 137.
‡ Leslie.
§ Leslie, Nat. Phil. I. 218.

from one place to another, would encounter the same resistance from friction, whether it performed the journey in one hour or in ten. The amount of friction, however, is augmented in very slow motions, and greatly diminished in those that are very swift. In this instance, the increase in the one case and the diminution in the other, appears to have some relation to the principle, that the friction of bodies is increased by their remaining in contact. From some observations of Professor Playfair made at the slide of Alpnach, where large fir trees are carried with great velocity down an inclined plane eight miles in length, it would appear that, in the case of very great velocities, friction is not, according to the common doctrine, either proportioned to the pressure or independent of the velocity; but that the ratio to the pressure is greatly diminished, and the actual resistance is far less than at common velocities. Thus, none but large trees could descend the plane at all; and when a tree broke into two pieces, the larger part would proceed while the smaller would stop; and the trees acquired in their descent a rapidity of motion, incompatible with the supposition that "friction acts as a uniformly retarding force," which has been considered as an established principle.*

The foregoing considerations are in favor of rapid travelling, whether on common roads or on railways, since the amount of the resistance is so much less than in slow movements; and accordingly it is said that the great speed given to stage coaches in England, amounting in some instances to 10 or 12 miles per hour, has not been attended with the degree of exhaustion to the teams that would have been anticipated.†

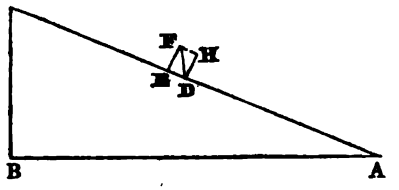
345. The angle at which an inclined plane must be elevated, in order that a given body resting on it may be on the point of sliding, is called the *angle of friction*, or sometimes the *angle of repose*. Let ABC (Fig. 166.) be the inclined plane, and let FD, parallel to BC, represent the whole weight of the body. FD being resolved into FE perpendicular to the plane, and ED coincident with it, ED will be the force with which the body tends to slide down the plane,

* Playfair's Works, Vol. I. or Ed. Phil. Jour. VI. 345. Also see Nicholson's Oper. Mech. II. 225.

† Nich. Oper. Mech.

and of course, that which is resisted by the friction, and hence it may be taken as the representative of the friction. But ED is to FE as BC to BA, or as the height of the plane to the base, or as

Fig. 166.



the tangent of the angle of friction to radius. Consequently, putting F for the friction, P for the whole pressure on the plane, and α for the angle of friction, then,

$$F : P :: \tan.\alpha : \text{rad.} \quad \therefore F = P \times \tan.\alpha;$$

that is, the friction is always such a part of the entire pressure, as is denoted by the product of that pressure into the tangent of the angle of friction. This angle often determines the figure which natural objects spontaneously assume. Hence, sand hills are more sloping than eminences composed of ordinary mould, the movable particles arranging themselves in obedience to the foregoing law. The angle of friction of iron pressing on iron, is found to be 16 degrees. The angle given to the threads of a screw, is regulated on this principle.*

346. The laws of friction in *rolling* bodies were ascertained by Coulomb, by comparing the forces necessary to roll a cylinder upon a table under various circumstances; and by similar experiments, were found the modes in which friction takes place in bodies *revolving* on an axis. The comparative loss of power which takes place in these three cases is as follows:

Friction of the sliding body equal to $\frac{1}{4}$th the pressure or 25 per cent.
do. revolving do. 15
do. rolling do. 5

In the case of hollow cylinders revolving on an axis, the *leverage* of the wheel aids in overcoming friction as has been already explained in Art. 300.

* Leslie.

347. *Friction wheels*, a contrivance by which friction is diminished in the greatest degree possible, owe their efficacy in part to the operation of the same principle.* Here the axis of a wheel, instead of revolving in a hollow cylinder, or instead of rubbing against a fixed surface, rests at each of its extremities, on the circumference of two wheels placed close by the side of each other, with their circumferences intersecting. The axis rests at the point of intersection, and as it revolves, the wheels revolve with it with the same velocity, and thus all friction between them and the axis is prevented, and what remains in the machine in consequence of the weight of the wheels themselves is *transferred* to the axles, and therefore (Art. 300.) is diminished, in the ratio of the diameter of one of the wheels to that of its axis. This combination may be repeated by several pairs of friction wheels. Eight wheels would contract the friction to the thousandth part.†

348. Other more common methods of diminishing friction are, by rendering the surfaces smooth, by using rollers, and by lubricating the parts in contact. The amount of friction in the several mechanical powers is very different. In the lever it is very small, especially when the turning edge is of hardened steel and shaped like a knife or prism, and turns upon a hard and smooth basis. The Wheel and Axle acting upon the same principle as the lever, occasions but little friction. The stiffness of the cordage, however, and the friction of the gudgeons‡ of the axis, have an effect in most cases equal to about 8 or 10 per cent. of the entire resistance. The Pulley is attended with great loss from this source. It is rarely less than 20 per cent. and often exceeds 60. The Inclined Plane involves but little friction when bodies simply roll on it; but when heavy bodies rest on axes, as in wheel carriages, the resistance from friction takes place in the same manner as upon plane surfaces. The transportation on inclined planes, as railways, is usually by means of wheels, since the resistance to sliding movements is too great to permit the use of them. The Screw is attended with a great deal

* A fine example of the application of the friction wheels is seen in Atwood's Machine.

† Leslie, El. Nat. Phil. I. 233.

‡ Pivots when large take the name of *gudgeons*.

of friction. Those with sharp threads have more than those with square threads, and the endless screw has most of all.* . In both the Screw and the Wedge, the friction evidently exceeds the resistance; otherwise they would not retain their position.

349. Friction is not, therefore, in all cases to be considered as unfavorable to the operation of machinery. It is, in many instances, a highly useful force. Many structures, as those of brick and stone, owe no small part of their stability to the roughness of the materials of which they are composed; without this resistance, the screw and the wedge would lose their efficacy, and wheels could not advance, nor could animals walk on the ground; and nails would lose their power of binding separate parts together. The art of polishing surfaces depends on the same cause, and the edges of most cutting instruments are saws, the teeth of which are more or less fine, and act on a similar principle. Even in certain rotary motions, Friction, by a rope or otherwise, becomes a moving force and urges a body in particular directions contrary to the force of gravity.

The resistance which moving bodies sustain from the air, and from water, will be considered hereafter.

CHAPTER VI.

OF PROJECTILES AND GUNNERY.

350. Early in the 17th century, Galileo, a celebrated Italian philosopher, established the mathematical theory of Projectiles, as given in the former part of this work. Since missiles thrown by instruments of warfare are among the number of projectiles, these principles were supposed to constitute the foundation of the art of Gunnery; and several of the kings of Europe, held out munificent encouragement to experiments in this department of science.† Galileo had indeed intimated that the *resistance of the air*, (which is not taken into the account in the mathematical theory of projectiles,)

* Emerson. † Robison's Mech. Phil. I. 167.

might occasion some deviation in bodies from the curve of a parabola; but this fluid appears so light in comparison with the heavy metals of which balls are made, being 10,000 times lighter than lead, that for a long time afterwards it was totally neglected.

351. About the year 1740, Mr. Robins, an eminent English mathematician, instituted a series of experiments on Gunnery, which showed that the parabolic theory of projectiles is so modified by the resistance of the air, as to be wholly inapplicable to practice. Experiments on the same subject have since been performed by Count Rumford, and by Dr. Hutton of the Royal Military Academy at Woolwich, which have confirmed the results obtained by Mr. Robins. By the labors of these several gentlemen, the parabolic theory of projectiles has received its proper modifications.

352. It is ascertained, in general, that projectiles moving *slowly*, describe curves which are nearly parabolas; while such as move *swiftly*, deviate very far from this curve. Indeed, the curve which a body moving with great velocity in the air describes, is so complicated, that the utmost resources of the calculus have hardly been able to investigate its nature;* and it is a remarkable fact, that we understand the motions of the heavenly bodies better than we do those of a cannon ball, and can trace the path of a planet better than we can that of an arrow. The parabolic figure described in the case of projectiles moving slowly, may be observed in tracing the path of a small stone thrown into the air, and more especially in the curves described by jets of water, spouting upwards, as in fountains. But when the jet of water is more rapid, and spouts at a high angle, as 45 degrees for example, we can plainly see that the curve deviates greatly from a parabola. The remote branch of the curve is seen to be much less sloping than the rising branch; and even in very great jets which are to be seen in some great water works, the falling branch is almost perpendicular at its remote extremity; and the highest point of the curve is far from being in the middle between the spout and the place where the water falls. This unequal division of the curve by its highest point may also be observed in the flight of an arrow or a bomb-shell.†

* Hutton's Tracts, No. 37. † Robison's Mech. Phil. I. 185.

353. The following facts also show the discordance between the parabolic theory of gunnery and experience. A cannon ball, fired in such a direction and with such a velocity, that its random or horizontal range, ought to be 24 miles, comes to the ground short of one mile. The times of rising and falling, if that theory held good, ought to be equal; but the time of rising is greater than that of falling at great elevations, and at small elevations, less than that of falling. According to the theory, the greatest random is at an angle of elevation of 45 degrees, (Art. 191.) but in practice it is found to be much below this. The greatest random of an arrow is when the elevation is about 36 or 38 degrees. Indeed, the angle for the greatest horizontal range, may be at all degrees from 45° to 30°; the slowest motions and the largest shot being almost at 45°, but gradually more and more below that degree as the shot is smaller and the velocity is greater; till at length with the most rapid motions, and the smallest shot, the angle is little above 30 degrees.* The following experiments were made in France by Borda, with a 24 pounder, with the same charge of powder in each experiment.

Elevation.	Range.
15°	1950
30	2235
45	2108
60	1700
75	950

Whence it appears that at the elevation of 15 and 75, the randoms instead of being the same, (being equally distant from 45,) were as the numbers 1950 and 950.†

354. All this discordance between theory and practice is owing to the resistance of the air, which, when the projectile moves with great velocity, becomes enormous. Nor will it be difficult on a little reflection, to comprehend the reason why this resistance should be so great. The force with which a projectile strikes the air at rest, is the same as that with which the air moving with equal velocity would strike the body at rest. This, in the case of a cannon ball, would greatly exceed the most violent hurricane. Again, as a ball moves through the air, it displaces, that is, gives motion to,

* Hutton, Tract 37. † Robison's Mech. Phil. 183.

great quantities of air; yet whatever motion it imparts to other bodies is extinguished in itself. The loss of motion therefore increases very fast with the velocity. It is said to be in general as the square of the velocity: so that a body moving through the air with ten times the velocity of another body, would encounter one hundred times as much resistance. In very swift motions, the resistance was ascertained by Robins to be even much greater than in the ratio of the square of the velocity.

355. The researches of Mr. Robins were made chiefly by the aid of an instrument of his own invention, called the *Ballistic Pendulum*. It consists of little more than a large block of wood, like a log, suspended after the manner of a pendulum. Now if a bullet be fired into the block, as the bullet will be stopped, and as it imparts to the block whatever motion it loses, consequently the momentum of the block after the stroke, is precisely that of the ball before the stroke. Hence the weight of the block and that of the ball being known, and the velocity imparted to the block being readily determined by observation, it is easy to find the velocity of the ball; for the weight of the ball is to the weight of the block, as the velocity of the block is to the velocity of the ball.*

Example.—On firing an 8 pound cannon shot into a ballistic pendulum that weighed 500 pounds, the pendulum was made to move over a space of 16 feet per second: What was the velocity of the ball? Ans. 1000 feet per second.

356. This simple apparatus is sufficient for ascertaining a great number of particulars relative to the art of gunnery. If the ball is fired nearly in contact with the block, we find with what velocity it leaves the gun; if at different distances from the block, we find how much the velocity is retarded by passing through the air, for those distances respectively. If at a given distance we vary the charge of powder, we find the respective changes which the velocity un-

* That is, putting W, w, for the weight of the block and ball respectively, and V, v, for their velocities, then $W \times V = w \times v$, and $w : W :: V : v$. In other experiments on this subject, the gun itself has sometimes been suspended like a pendulum, and the velocity of the ball estimated by the distance to which the gun recoiled, since action and reaction are equal, and in opposite directions.

dergoes, and hence learn the ratio that ought to be observed between the powder and the ball, in order to produce the maximum effect. The effects resulting from variations in the length, shape and bore of the gun, are also ascertained with equal facility.

357. The following are some of the practical results ascertained by the experiments of Mr. Robins, Count Rumford, and Dr. Hutton. A musket ball, discharged with a common charge of powder, issues from the muzzle of the piece with a velocity between 1600 and 1700 feet in a second.* The utmost velocity that can be given to a cannon ball is 2000 feet per second, and this it has only at the moment of leaving the gun. In order to increase the velocity from 1600 to 2000 it requires half as much more powder, which involves a hazardous strain upon the gun, and the velocity will be reduced to 1300 before the ball has proceeded 500 yards.†

358. Mr. Robins, in the course of his experiments on the resistance of the air, made a curious observation. In very moderate velocities, the retardations were nearly as their squares. As the velocities were increased, the resistances increased at a somewhat greater rate, but with a certain observable regularity, till the velocity exceeded 1100 feet per second. But when the velocity is increased from 1100 to 1200, the increase of resistance is prodigious. After this, the resistance goes on increasing nearly with its former regularity. Mr. Robins accounts for this singular fact in the following manner. As the ball rushes through the air, the air falls in behind it, being pressed in by the weight of the surrounding air. But the ball may move so rapidly that the air cannot instantaneously fill up the place left by the ball. In this case the ball is retarded, not only by the resistance of the air which it displaces, but also by its having the pressure of the atmosphere on one side and not on the other. It is found by calculation, that the velocity with which air rushes into a vacuum, is about 1100 or 1200‡ feet per second; conse-

* Space fallen through to acquire the velocity of 1600 feet per second $=\frac{(1600)^2}{64\frac{1}{3}}$. (Art. 34, p. 27.) $=40.000$ *nearly* $=7.6$ miles.

† Robison's Mech. Phil. I. 201.

‡ The velocity with which air *begins* to rush into a void, has been estimated at 1338 feet. (See Cambridge Mechanics, p. 377.)

quently, it is when the ball is moving at this rate, that it loses the entire pressure of the atmosphere behind it.* It is remarkable (says Dr. Robison,) that this is also nearly the velocity of sound, and there is an observation of this kind, which seems to have some connection with the mechanical fact observed by Mr. Robins. If a person stand in such a direction from a cannon, when it is discharged, that the ball may pass him at no great distance, he will hear the noise made by the ball rushing through the air at the time of its flight, and as the ball approaches him, the noise should become more audible. But he will hear the noise loudest at the very first, immediately following the report of the gun; and after about two seconds, he may observe the sound change all at once, and not only become more faint, but even change its kind, after which the sound increases as the ball comes nearer. It seems highly probable that this abrupt alteration in the sound, takes place just at the time when the resistance undergoes such a change; and that it is owing to the difference in the nature of the undulations, when there is a void behind the ball, and when there is not.†

359. From the foregoing considerations it is inferred that great charges of powder are absolutely useless in the service of artillery, especially when the distance of the object is considerable, and that a velocity exceeding 1100 should not be aimed at. The maximum service charge is $\frac{2}{3}$ the weight of the ball. In close naval engagements great velocities are injurious, for the ball may then pass through both sides of the vessel without lodging, and the number of splinters produced by a ball in rapid motion, is much less than is caused by one moving more slowly. By reducing the charge we may also reduce the size and strength of the gun; and hence guns are made of smaller dimensions now than formerly, in order to do the same execution. The velocity with which a charge of powder expands itself at first, is estimated by Hutton as high as 5000 feet per second.‡ As it expands, this velocity is of course constantly diminishing, but will exceed that of the ball while the latter is passing through the barrel of the gun, and will act as a constantly accelerating force. Long guns therefore give to balls a greater velocity than short ones;

* Robins, Tract on Gunnery, p. 181.

† Robison's Mech. Phil. I. 200. ‡ Hutton, Tract 37.

but the gain secured in this way after a moderate length is so small,* (there being also some disadvantages peculiar to long guns,) that cannon have of late years been much shortened. In the naval service, *carronades* have been introduced. These are a short kind of gun, with small bore, requiring for a charge of powder, only one twelfth the weight of the ball. Their weight and thickness are proportionally reduced, yet in close action they produce effects superior to those of long guns.†

360. It has been found that no difference is caused in the velocity or range, by varying the weight of the gun, nor by the use of wads, nor by different degrees of ramming, nor by firing the charge of powder in several places at the same time; but that a very great difference in the velocity arises from a small degree in the *windage*.‡ Indeed with the usual established windage only, viz. about $\frac{1}{12}$ of the calibre, no less than between $\frac{1}{4}$ and $\frac{1}{3}$ of the powder escapes and is lost, and as the balls are often smaller than the regulated size, it frequently happens that half the powder is lost by unnecessary windage.§ To this cause also, namely, too great windage, Dr. Hutton ascribes a great part of the sideways deviation of a ball; since when in passing through the barrel of the gun, it is knocked from side to side, it will finally take the last direction which it happened to have at the muzzle of the gun.‖ Another cause of this deviation from the line of direction, arises from a want of perfect sphericity in the ball, by which means the two sides do not meet with equal resistance. Rifles owe their superiority over common guns, chiefly to their obviating this deviation. They have a spiral groove cut in their bore, making about a turn and a half in the whole length of the barrel. The ball, which is made to fit close to avoid too great windage, has a corresponding motion impressed on it, which it retains after it leaves the gun, continuing to revolve around the line of direction. Whatever inequalities, therefore, may exist in the ball, their effects are

* The random, according to Dr. Hutton, increases only as the fifth root of the length, which is so small an increase as to amount to only about a seventh part more range for a double length of gun. (Hutton, Tract 37.)

† Renwick, Heads of Lectures, in Literary and Scientific Repos. IV. 281.

‡ By windage is meant, the difference between the diameter of the ball and that of the bore of the gun.

§ Hutton, Tracts 3. 255. ‖ Hutton, Tract 38. Prob. 4.

neutralized, by their being first on one side and then on the other of this line.

361. When a ball is projected from a piece of ordnance, at a small angle of elevation, and falls upon water, or on a plane of hard earth, its flight will not cease, but it will rise again and describe a second curve similar to the first but less; and it will continue to rebound, until the whole of its projectile velocity is destroyed. This species of firing is called *Ricochet*. It is applied with great advantage from sea coast batteries upon shipping, and in the attack of fortresses. The pieces are fired with small charges of powder and elevated only from 3 to 6 degrees. The word signifies *duck and drake*, or rebounding; because the ball or shot thus discharged, goes bounding and rolling along, killing or destroying every thing in its way, like the bounding of a flat stone along the surface of water when thrown almost horizontally.*

CHAPTER VII.

APPLICATIONS OF THE PENDULUM.

362. THE pendulum has three most important and interesting uses, viz. as affording a *measure of time*, the means of determining the *figure of the earth*, and a *standard of weights and measures*.

363. Time is any portion of indefinite duration, from which it may be considered as separated, as a given number of yards of cloth are measured off from a piece of unknown length. For the measure of time we may employ any instruments or marks that divide it into equal portions. The period occupied by a given quantity of sand in running through a funnel, as in the hour-glass—the successive intervals through which the surface of a column of water descends while discharging itself by an aperture in the bottom or side of the vessel—the beats of the hand, as in music, or even the pulsations

* Hutton's Math. and Phil. Dict. II. 374.

of the wrist, are so many different modes of measuring time. But as the intervals measured by any of these modes are not perfectly uniform, and as they are incapable of that minute subdivision which is sometimes required, none of them is adapted to the purposes of the astronomer, who seeks by this method to estimate the motions of the heavenly bodies, or of the mariner who depends on his time-piece for the means of ascertaining his longitude at sea.

364. The adaptation of the pendulum to the measuring of time, was first noticed by Galileo on observing the vibrations of a lamp suspended from the ceiling of a church. The theoretical doctrines of the pendulum revealed the fact, that all the vibrations of a pendulum, of given length, are performed in nearly equal times, and, when made to move in the arc of a cycloid, (Art. 180.) in times that are exactly equal. Hence the pendulum becomes a measure of time. Galileo, indeed, used this instrument by itself, counting the number of its vibrations; but Huygens, an astronomer of Holland, first connected it with clock-work, by which its motions are indefinitely continued, more precisely regulated, and the number of its vibrations exactly registered. Pendulums are usually made of such a length as to beat seconds, or to indicate, at each vibration, the $\frac{1}{86400}$ part of a mean solar day; but as the period occupied by the passage of a star from the meridian round to the same meridian again (called a siderial day) is less than that occupied by the sun, so the pendulum, regulated to solar time, may, by being shortened a little, be made to indicate the corresponding aliquot parts of a siderial day. Since, moreover, the times of vibration are as the square roots of the length, (Art. 181.) a pendulum may be made to beat half-seconds by being made only one fourth the length of the seconds pendulum, or to beat once in two seconds by making it four times as long.

365. As the vibrations of the pendulum in circular arcs are not exactly equal to each other, it has been attempted to make it vibrate in the arc of a cycloid (Art. 179.); but the practical difficulties involved in making the pendulum rod adapt itself to the cheeks of a cycloid are so great, that this method has been abandoned in practice, and is at present regarded only as a theoretical curiosity. Where the arcs of vibration are very small, they will not differ sen-

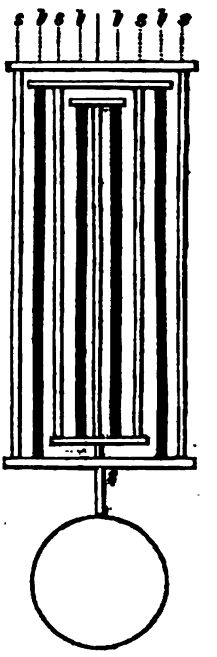
sibly from portions of a cycloid, and the times of vibration may without sensible error be considered as equal. In cases where extreme accuracy is required, the exact allowance, mathematically determined, may be applied to vibrations performed in circular arcs to reduce them to the corresponding vibrations in a cycloid; and a still further allowance may be made for the resistance of the air, so as to make the vibrations correspond to such as they would be if performed in a vacuum. The pendulum is still liable to some sources of inaccuracy, which it has cost much labor and skill to obviate. The rod and bulb are both subject to expand by heat and contract by cold. In the former case the center of oscillation (Art. 170.) is carried too far from the center of motion, that is, the pendulum becomes too long, and the clock goes too slow; in the latter case, the pendulum being shortened, the clock goes too fast. With a pendulum having an iron rod, a difference of temperature of 25 degrees, would make a difference of six seconds in 24 hours in the rate of the clock.*

Pendulums so constructed as to remedy these defects arising from change of temperature, are called *Compensation Pendulums.* The general principle of these instruments is as follows: we connect with the pendulum rod, some substances which are made to expand or contract in a direction opposite to that in which the rod itself contracts or expands, and thus maintains the center of oscillation, at a uniform distance from the center of suspension. The length of the theoretical or simple pendulum, it may be remarked, depends wholly on the distance between these two points, and not upon the length of the pendulum rod. If then we make the pendulum rod of one kind of metal as steel, and connect with the bulb another kind of metal as brass, which is expanded more by the same amount of heat, a less length of the latter expanding upwards, will compensate for a greater length of the former expanding downwards, and the center of oscillation will be kept at the same constant distance from the center of suspension. Now it is found by experiment that the lengths of these substances which are equivalent to each other are as 1 : .6091, or as 100 : 61 nearly. Therefore, if we connect 61 inches of brass with 100 inches of steel, the former so arranged as

* Kater.

to expand upwards while the latter expands downwards, we shall have a compensation pendulum. The first instrument constructed on this principle is called the *Gridiron Pendulum*, a name which it derives from the fancied resemblance which the parallel bars of steel and brass have to the gridiron. In the following figure, the five bars marked *s* are of steel, and the four marked *b* are of brass. It will be seen by the figure that the rods of steel can elongate themselves only downwards, while those of brass can expand only upwards; and being combined in the ratio of 100 to 61, they will exactly compensate each other. A clock furnished with the gridiron pendulum is found capable of keeping very accurate time. Indeed a clock of this kind, constructed by Harrison, the inventor of the gridiron pendulum, gained the great premium offered by the British Board of Longitude, for a time-keeper which would keep time for a given period to a certain degree of accuracy.

Fig. 167.



Another form of the compensation pendulum consists of a jar of mercury, suspended from the rod in the place of the bulb. By means of a kind of stirrup the jar is so connected with the rod as to expand upwards while the rod expands downwards. The mercury being a very expansible metal, the lengths required for mutual compensation are as 1 : .07; that is, 39 inches being the length of the steel rod, the length of the column of mercury is only $39 \times \frac{7}{100} = 2.73$ inches.

366. The use of the pendulum *in investigating the figure of the earth*, results from its power of measuring the intensity of the force of gravity in any given place. (Art. 182.) Now as this force varies inversely as the square of the distance from the center of the earth, when the force of gravity is ascertained at a great number of places remote from each other, a comparison of these observations indicates the respective distances of these places from the center of the earth, and of course, when the observations are sufficiently multiplied, they indicate, collectively, the figure of the earth.

Since in this mode of ascertaining the force of gravity, the *length* of the pendulum is a given quantity, and since that length varies at different places, it is necessary to ascertain, by experiment, the true length of the seconds pendulum for each place of observation. The most convenient method of doing this is that of Kater. It depends on the principle that the centers of suspension and of oscillation are convertible points; that is, instead of making the pendulum vibrate in the ordinary manner, we may exchange ends and cause it to vibrate on the center of oscillation, while the place of the latter is occupied by what was before the center of suspension. Hence, the pendulum being so constructed that it may be suspended from either end, two knife edges on which, respectively, the pendulum is to vibrate being placed at *nearly* the required distance from each other, a slide is attached to the intermediate rod by moving which one way or the other, the pendulum may be made to vibrate precisely the same number of times, in a given period, whether suspended at one end or the other.

367. It is not necessary, however, to vary the length of the experimental pendulum, in order to ascertain the true length of the seconds pendulum at different stations. These lengths may be ascertained, respectively, by comparing the number of vibrations made by the same pendulum at the several places. For (Art. 181.) the time of vibration is as the square root of the length, or $T \propto \sqrt{L}$; but the time is inversely as the number of vibrations, or $T \propto \frac{1}{N}$. Hence $\sqrt{L} \propto \frac{1}{N}$ and $L \propto \frac{1}{N^2}$. Hence, calling the length of the experimental pendulum, at any other station L′, and we have

$$L : L' :: \frac{1}{N^2} : \frac{1}{N'^2}, \therefore L' = L \times \frac{N^2}{N'^2}; \text{ that is,}$$

Multiply the length of the experimental pendulum, by the square of the number of vibrations, at any given place, and divide the product by the square of the number of vibrations at any other place, and the quotient is the length of the pendulum at the latter place. If the experimental pendulum is adjusted to seconds, this rule will give the lengths of the seconds pendulum at the several places respectively.

368. The result of the observations made with the pendulum, like those obtained by measuring arcs of the meridian, show that the figure of the earth is that of an oblate spheroid, having the polar diameter shorter than the equatorial. The most correct determination is supposed to be $\frac{1}{299}$ of the greater diameter; that is, the polar is so much shorter than the equatorial diameter. This is denominated the *ellipticity* of the earth. From the measurements of arcs of the meridian, the ellipticity of the earth is found to be $\frac{1}{310}$, making a difference of about twenty six miles in the two diameters.*

369. A third important application of the pendulum is, *as a standard of linear measures.*

In order to insure confidence in business transactions, it is very essential that the weights and measures employed, should be and remain of a certain known amount or length,—a condition which cannot be attained otherwise than by adapting them to a fixed and invariable standard. To fix on such a standard, is a matter of more difficulty than would at first be imagined. All things on the face of the earth are more or less subject to change, not only the works of man, but even those of nature. The elements decompose or wear down, in time, the hardest rocks. In ancient times, the standards of weights and measures were derived from the parts of the human body; as, a *hand's breadth*, a *cubit*, a *digit*, &c. The English linear measures were for a long time, referred to the length of a yard-stick taken from the length of the arm of Henry the VIIth, and preserved in the tower of London.† Of the defectiveness of such a standard, it is sufficient to mention the impossibility of verifying it after the death of the king. The idea of standards involves two conditions that are indispensable,—the constancy of the thing itself, and the power of verification, should any change in it even be suspected. These conditions are secured by connecting the standard with the immutable laws of nature. In the year 1790, the French government undertook to effect a complete change in the weights and measures in use throughout the world, and to derive an unalterable standard from the dimensions of the earth itself. For this purpose, they undertook the determination of the exact length of a

* Herschel's Astron. p. 114. † Adams on Weights and Measures.

quadrant of the meridian, extending from the equator to the North Pole. A certain aliquot part (the ten millionth) of this they denominated a *metre*, which was to be the unit of all *linear* measures. The square of this furnished a measure of *surfaces*, and the cube a measure of *solids*. Measures of *weight* were derived from the weight of a certain volume of water.

370. In the French system, the whole of the divisions and multiples of the units were decimals, which were named by prefixing the Greek numerals to the multiples, and the Roman numerals to the decimal subdivisions of the units. Thus the measures of length are,

Myriametre,	= 10000	metres.
Kilometre,	= 1000	"
Hectometre,	= 100	"
Decametre,	= 10	"
Metre,	= 1	"
Decimetre,	= $\frac{1}{10}$	"
Centimetre,	= $\frac{1}{100}$	"
Millimetre,	= $\frac{1}{1000}$	"

Upon this system Professor Renwick remarks as follows,* "No system can be imagined more perfect and beautiful, in a scientific point of view, than this system of the French nation. It is founded upon a standard existing in nature, and invariable, and which is susceptible of determination to such a degree of exactitude, that no probable error that can, in the present state of science, be committed in the measure of degrees, will affect the small fraction of the standard that forms the unit of length. From this decimal division, it is exactly consistent with our usual system of arithmetic; and its nomenclature is systematic, and of easy recollection. Still it is not without fault, even in a scientific point of view. The measures of length are incapable, for instance, of application to astronomical purposes, in which we use the semidiameter of the earth, and not its quadrant as the unit; and these two magnitudes are incommensurable. Neither are we aware that a measure of the meridian in other countries particular in our own hemisphere, would reproduce the same magnitude for the quadrant that was obtained in France. The

* Mechanics, p. 296.

measure of a sufficient arc, whence to determine the length of a quadrant, is an operation of great expense, and would occupy a long time. Hence, in presenting the types of the units to the National Assembly, the commission propose to verify them, if suspected of alteration, and reproduce them, if lost, by reference to the pendulum of the Observatory of Paris; thus recurring to the same natural standard that had been rejected by them in the outset. The metre is, therefore, after all the labor that was expended in its determination, no more than a conventional length, whose relation to the seconds pendulum of a particular place is well determined. It has also been found impracticable to introduce the decimal division into the measure of angles; and after strenuous attempts for that purpose, and the laborious construction of new tables, even the astronomers of France have returned to the ancient division of the circle.

371. "The objections, in a practical point of view, are more numerous, and have been found insuperable. Thus, however well calculated for scientific purposes, and even for those of commerce, the decimal multiples of the unit may be, decimal subdivisions have been found unsuited for the purposes of retail traffic; for this object no other than a binary system can, with convenience, be used. In fact, in the subdivisions of the unit, no other method appears to be consistent with nature; and those systems which are founded on division by two, appear to defy any attempts to alter them. Thus the system of money in the United States, which is strictly decimal, is only used in written calculations; while the old binary division of the Spanish dollar is retained in all retail operations, in spite of the barbarous nomenclature that is applied to it in some of the States. Several of the units of the French system, or their decimal divisions and multiples, are unsuited to ordinary transactions; subdivisions suitable to these were, therefore, first introduced clandestinely, and afterwards sanctioned by law. Thus a measure of the length of about a foot, is the most convenient for many mechanical uses; and for this purpose a measure of the third part of a metre was formed, called the Metrical Foot, to which the ancient duodecimal subdivision was applied. The kilogramme differing but little from two ancient pounds, its half has become the unit of weight in actual use, and is called the Metrical Pound; to this also, a binary division has been applied,

and the decimal system in the descending scale has not only failed in being introduced into commerce, but has been abolished by authority.

372. "Thus there are at present in France, in fact, three diverse systems; the ancient, which is not wholly abandoned; the decimal system of the commission; and a system derived from the latter, to which the ancient names, and many of the ancient subdivisions are applied.

"The system proposed, and partially introduced by the French philosophers, may, therefore, be considered as a splendid failure, worthy however of a better fate, from the scientific skill with which the operations connected with it were executed. It is also memorable for the light it has thrown on all analogous processes, and the actual benefit the researches, in respect to it, have conferred upon physical science. Warned by the example of the French, the British, Danish, and American governments, have wisely restricted themselves to the verification of the measures in actual use, and their restoration to their true dimensions. The two former, and the state government of New York, have referred them to the pendulum, a standard existing in nature, determinate, and easily verified."

373. We may exemplify the precautions necessary in order to make the pendulum an accurate standard of measures by reviewing the conditions under which it is adopted as a standard in the state of New York.

1. The standard is the pendulum vibrating seconds, in a cycloidal arc, and in a vacuum in Columbia College in the city of New York. The vibrations are required to be in a *cycloidal* arc, because those performed in circular arcs are not absolutely uniform. But the vibrations performed in circular arcs may be *reduced* to the corresponding cycloidal arcs upon mathematical principles. As the resistance of the air might be unequal at different times, such an allowance must always be made for this, as will neutralize its effect. A particular spot is designated, because local causes have some influence upon the vibrations of the pendulum.

2. The unit of linear measures is the *yard*, which is of such magnitude as to bear to the pendulum the proportion of 1 to 1.086158. Some have proposed to make the seconds pendulum itself the unit of linear measures, and to make all other measures of length parts and multiples of this; but it is thought advisable to retain the common measures to which the habits of society are adapted, determining their exact amount by referring them to this invariable standard. The usual subdivisions of the yard into feet, inches, and so on, remain as they are. The standard *temperature* of the standard yard-stick, is that of melting ice. It is necessary to attend to this circumstance, because all bodies are expanded by heat and contracted by cold, so that the yard-stick is of a uniform length only at a given temperature.

3. The unit of measures of *weight* is the Avoirdupois pound, of such magnitude that a cubic foot of pure water, at its maximum density* shall weigh 1000 ounces or 62½ pounds.

4. The unit of dry measures of *capacity* is the gallon, a vessel of such magnitude as to hold exactly 10 lbs. of pure water, at its maximum density. The bushel therefore holds 80 lbs. The unit of liquid measures is also a gallon, containing eight pounds of distilled *water*, at its maximum density.†

* Water expands as its temperature rises above or falls below a certain point; about 40° of Fahrenheit. Hence the necessity of employing it as a standard at its *maximum density*.

† See an able view of the applications of the Pendulum in Renwick's Mechanics.

PART III.—HYDROSTATICS.

374. The principles of Mechanics demonstrated and explained in the foregoing pages, are *universal* in their application, extending alike to all bodies, whether solid or fluid. But in addition to those properties which fluids have in common with solids, and which bring them under the general laws of Mechanics, they have also properties peculiar to themselves, and which give rise to a distinct class of mechanical principles, not applicable to solid bodies. These are embraced under the heads of Hydrostatics and Pneumatics, the former division comprising the doctrine of liquids, and the latter that of aëriform bodies or gases.*

375. In Mechanics, after having ascertained a few fundamental principles by experiment and observation, the superstructure is raised chiefly by mathematical reasoning, and thus the great body of truths in that science are established; but in Hydrostatics, and the other subjects of Natural Philosophy which follow, we are much more dependent on *experiment*, which frequently affords us more satisfactory evidence, than we can obtain by the application of abstract mechanical principles.

376. *A* fluid *is a body whose particles move easily among themselves, and yield to the least force impressed; and which, when that force is removed, recovers its previous state.*†

* In some treatises these subjects are distributed under the heads of Hydrostatics and Hydrodynamics, the former comprehending the mechanical properties of fluids *at rest*, and the latter those of fluids *in motion*. In other works, what relates to liquids or non-elastic fluids, is divided into *Hydrostatics* and *Hydraulics*, (the latter denoting the mechanical powers and agencies of running water and of machines carried by water,) and *Pneumatics*. The most scientific division is that adopted in the Edinburgh Encyclopœdia, where the term *Hydrodynamics* (from Τδωρ and Δυναμις) is used to denote in general the mechanical powers and agencies of fluids; and this head is divided into the two, Hydrostatics and Hydraulics. Pneumatics is treated of in a separate article. But we prefer to follow the example of those who arrange these subjects under the two heads specified in the text.

† Vince.

In accordance with the example of Sir Isaac Newton, which has been followed by most writers on Hydrostatics, we have introduced the foregoing definition of a fluid. Although the best perhaps that the subject admits of, yet it is not very discriminating, since such substances as quicksand, or the powder of magnesia, have their particles as movable as those of tar or syrup, while yet the former are solid and the latter fluid bodies. The fact is, that no definition can be given of a fluid which would convey to one unacquainted with these bodies any adequate idea of the distinction between them; and it may be doubtful whether any better definition can be given of a fluid, than that *it is a generic term, comprehending liquids and gases*, while liquids are defined to be *bodies in the form of water*, and gases *bodies in the form of air*. The property included in the definition of Professor Vince, which we have copied, namely, that a fluid when disturbed by any force impressed *recovers itself*, is as characteristic of this class of bodies as the mobility of their parts.

Since water, wind, and steam, are the only fluids that are usually employed as mechanical agents, the doctrines of Hydrostatics and Pneumatics, have regard chiefly to them; but the principles established respecting these, are applicable also to all analogous bodies.

377. It has been usual to denominate liquids and gases respectively *non-elastic* and *elastic* fluids, on the supposition that water and other liquids are nearly or quite incompressible. An experiment performed by the Florentine academicians, as long ago as 1650, seemed to prove that water is wholly incompressible. They filled a hollow ball of gold with water, and subjected it to a strong pressure.* The water, not yielding to the compression, oozed through the pores of the gold. Considering the great density and compactness of this metal, the experiment was for a long time held as proving decisively that water is wholly incompressible. Although this experiment shows that water is compressed with great difficulty, yet later experiments have proved, that it is still capable of compression.

* The method of applying the pressure is said to have been by means of a screw working through a water-tight joint.—*Partington*. As the screw was forced into the water, the la[illegible]must either be compressed or make its escape from the ball. (Art. 2.) An easier mode of applying the pressure would have been, to put the ball into a vise and flatten it, by which its capacity would have been diminished.

The most decisive evidence of this point has been recently afforded by the experiments of Mr. Perkins. It had been previously ascertained that by a pressure equivalent to that of the atmosphere, or about fifteen pounds to the square inch, water is compressed about one part in twenty two thousand. Mr. Perkins, by methods to be described hereafter, applied successive degrees of pressure up to that of two thousand atmospheres, or 30,000 pounds to the square inch, and found the contraction of volume to be nearly *one twelfth* of the whole.

With these preliminary remarks, we may now enter upon the immediate consideration of the principal subject before us.

CHAPTER I.

OF LIQUIDS OR NON-ELASTIC FLUIDS AT REST OR IN EQUILIBRIUM.

378. HYDROSTATICS *is that branch of Natural Philosophy which treats of the mechanical properties and agencies of* LIQUIDS.

379. *Fluids at rest press equally in all directions.*

A point in a mass of fluid, taken at any depth, exerts and sustains the same pressure in all directions, upwards, downwards, or laterally. This is the most remarkable property of fluids, and is what particularly distinguishes them from solids, which press only downwards, or in the direction of gravity. This property naturally results from the freedom of motion that subsists between the particles of fluids; for if, when a fluid is at rest, the pressure on any given portion were not equal in all directions, that portion would move in the direction in which the resistance was least. But by the supposition it does not move: therefore it is kept at rest by equal and contrary forces acting on all sides. But the most satisfactory evidence of this truth is obtained from experiments. On opening an orifice in the side of a vessel of water, and estimating the force with which the water issues, it is found to be equal to the weigh[redacted]he incumbent fluid; and the upward pressure of water at a certain depth is found

to sustain the heaviest bodies when exposed to its action alone, the column above the bodies, and of course the downward pressure being removed.

380. *A given pressure or blow impressed on any portion of a mass of water confined in a vessel, is distributed equally through all parts of the mass.*

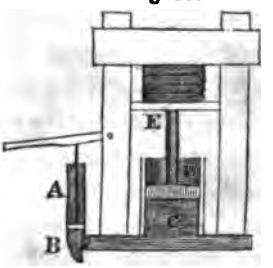
A given pressure, as that made by a plug forced inwards upon a square inch of the surface of a fluid confined in a vessel, is suddenly communicated to every square inch of the vessel's surface, however large, and to every inch of the surface of any body immersed in it. Thus if I attempt to force a cork into a vessel full of water, the pressure will be felt not merely by the portion of the water directly in the range of the cork, but by all parts of the mass alike; and the liability of the body to break, supposing it to be of uniform strength throughout, will be as great in one place as another, and it will break at the point where it happens to be the weakest, however that point may be situated relatively to the place where the cork is applied; and the effect will be the same whether the stopper be inserted at the top, the bottom, or the side of the vessel.

381. It is this principle which operates with such astonishing effect in the *Hydrostatic Press*, by means of which a single man can exert a force which is adequate to crush the hardest substances, or to cut in two the largest bars of iron.* Its construction is as follows. Fig. 168, represents a press made of the strongest timbers, the foundation of which is commonly laid in solid masonry. AB is a small cylinder in which moves the piston of a forcing pump, and CD is a large cylinder in which also moves a piston, having the upper end of its rod pressing against a movable plank E, between which and the large beam above is placed the substance to be subjected to pressure, as for example a pile of new bound books. By the action of the pump handle, water is raised into the small cylinder, and, on depressing the piston, it is forced through a valve at B into the larger cylinder and raises the piston D, which expends its whole force on the bodies confined at E.

* Partington's Manual of Nat. Phil.

Now since, whatever force is applied to any one portion of the fluid, extends alike to every part, therefore the force which is exerted by the pump upon the smaller column, is transmitted unimpaired to every inch of the larger column, and therefore tends to raise the movable plank E with a force as much greater, in the aggregate, than that impressed upon the surface of the smaller, as

Fig. 168.



this surface is smaller than that of the larger column; or (which is the same thing) as the number of square inches in the end of the piston B is less than that of the piston D. The power of such a machine is enormously great; for, supposing the hand to be applied at the end of the handle, with a force of only ten pounds, and that this handle or lever is so constructed as to multiply that force but five times, the force with which the smaller piston will descend will be equal to 50 lbs.; and let us suppose that the head of the larger piston contains the smaller 50 times, then the force exerted to raise the press-board will equal 2500 lbs. A man can indeed easily exert many times the force supposed, and can therefore exert a force upon the substance under pressure, equal to many tons.

The hydrostatic press involves far less loss from friction than any other species of press, and it is said that the naked force of man is more effective when applied in this way than in any other. By the mere weight of a man's body, when leaning on the extremity of the lever, a pressure may be produced of upwards of 2000 tons. It is the simplest and most easily applicable of all contrivances for increasing human power; and the only limit to the force which may be called into action by it, is the want of materials of sufficient strength to enable us to apply the enormous pressure which it generates.*

* Mosely, *Mech. applied to the Arts*, Art. 197.

382. The rationale of the principle of the Hydrostatic Press, will be best understood by recurring to the following principles,—that opposite forces are in equilibrium when their *momenta* are equal; that a small power may be made to balance a great weight, by making it move, in a given time, over a space as much greater than the larger does, as its weight is smaller; and that it may be made to overcome that resistance or weight and give motion to it, if its velocity be greater than that of the latter in a still higher ratio. Now to apply these principles to the case before us, it is evident that any quantity of water forced out of the smaller into the larger cylinder, must rise in the latter as much slower as the area of the horizontal section is larger. If, for example, the capacity of the larger cylinder were ten times that of the smaller, then a quantity of water one inch in height transferred from the smaller to the greater cylinder, would occupy only the height of one tenth of an inch, consequently, the depression of the small piston one inch would raise the large one only the tenth of an inch. This case therefore resolves itself into that general principle, according to which a vast force is exerted through a short distance, by moving a small force through a distance much greater.

This press is used for the extraction of oils, either vegetable or animal, for pressing paper or books, and for packing cotton and other substances. Hay intended as food for cattle on ship-board, is reduced by this press almost to the state of a solid, and enormous quantities are thus brought into an inconceivably small compass. There would seem to be no force known to us which may not be made to yield to this power. It requires but one of its least efforts to tear up a tree by its roots, or to break a large beam asunder.

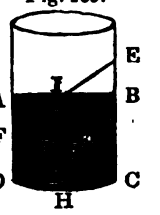
383. *The surface of a fluid at rest is horizontal.*

The evidence of the truth of this proposition is threefold.* *First*, this result is a natural consequence of the mobility of fluids, since, if any portion is raised above the rest, having nothing to support it, and being acted on by gravity, it must descend in the same manner as a

* It might appear superfluous to offer so much proof of a point so plain; but the several modes of reasoning will serve to instruct the young learner in the peculiar properties of fluids.

body placed on a perfectly smooth inclined plane. *Secondly*, whenever a body is free to move, its *center of gravity* will descend as low as possible. (Art. 70.) When, therefore, any portion of a fluid is raised above the general level, the center of gravity of the mass is raised, and drawn out of the line which passed perpendicularly through it and the point of suspension, (Fig. 43.) and it must return to that line before the fluid can be at rest. Thus, let ABCD be a body of water, contained in a cylindrical vessel; let I be the center of the surface, and IGH the perpendicular line passing through the center of gravity G and the base. Through I, suppose a plane, movable on a hinge at E, to pass, and to be depressed into the situation FIE, by which means the water will be depressed on the side AF and raised on the side EB, while the center of gravity will be removed to the point G′. Now let the plane be removed, and the center of gravity being free to move, it will vibrate around I as a point of suspension, until it finally recovers its situation at G, and the surface of the fluid will return to its original level. *Thirdly*, experience shows that the proposition is true, since fluids, when free to move, always settle themselves with their surfaces parallel to the horizon.* It must be understood, however, that the surface of large bodies of water, is not, strictly speaking, a horizontal level, but is a portion of the convex surface of the earth; for since the center of gravity of every portion of the fluid will descend as low as possible, the whole will dispose itself around the center of attraction so as to form a portion of the earth's surface. For small distances the curvature is so slight that it may be neglected, not amounting to one second of a degree for 100 feet;† and for the distance of a mile, the deviation from a straight

Fig. 169.

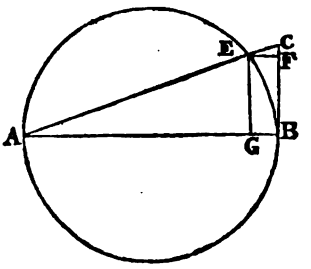


* The only exceptions to this law, are those arising from the *attraction of cohesion* among the particles of liquids, and the attraction exerted by small tubes, called *capillary attraction*. In consequence of cohesion, small portions of a liquid form themselves into drops, and large portions present a convex surface, as is strikingly exhibited in a wine glass filled with quicksilver. By capillary attraction, the surfaces of liquids are made concave, a phenomenon to which we shall attend more particularly hereafter. But both these causes operate on a scale comparatively very small.

† $\frac{360\times60\times60\times100}{25000\times5280} = \frac{1296}{1320} = \frac{9'}{10}$.

line drawn in the direction of a tangent, is not more than 8 inches. The amount of the depressions for different distances, is estimated by the following formula, $D=\frac{2L^2}{3}$ where L represents the number of miles, and D the depression in feet. Thus the depression for 2 miles

Fig. 170.



is $\frac{2\times4}{3}=2\frac{2}{3}$ feet. Let FE or its equal GB be the depression for the distance BE. For moderate distances this arc may be considered as equal to its chord. Hence, GB(D) : EB(L)::EB : AB; or $D=\frac{L^2}{AB}$, or, D being expressed in feet and L and AB in miles, $D=\frac{L^2\times(5280)^2}{AB\times5280}=\frac{L^2\times5280}{7912}=\frac{2L^2}{3}$, nearly.

1 Mile. - - - -	$\frac{2\times1^2}{3}$	=	$\frac{2}{3}$ feet.
2 " - - - -	$\frac{2\times2^2}{3}$	=	$2\frac{2}{3}$ do.
3 " - - - -	$\frac{2\times3^2}{3}$	=	6 do.
100 " - - - -	$\frac{2\times100^2}{3}$	=	$6666\frac{2}{3}$ do.

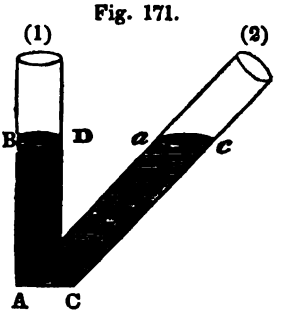
384. A practical application of this principle is made in the art of *levelling*. A level is sometimes made by merely cutting a groove or channel in a flat piece of board and filling it with water. When the board is brought into such a situation that the water in the groove remains stationary, the position is horizontal. But the *spirit level* is the instrument more commonly employed for this purpose. This consists of a small cylindrical tube of glass, from two to six inches long, filled with spirits of wine or ether, except a small space which

is occupied by a movable bubble of air. When such a tube is placed horizontally, the bubble of air will remain stationary in the center of the tube, at a fixed mark; but whenever the tube is inclined, in the least degree, the bubble will ascend towards the elevated end. Spirit levels are much used for adjusting astronomical, surveying, and other delicate instruments.

385. *The pressure upon any particle of a fluid of uniform density, is proportioned to its depth below the surface.*

Case 1. Let the column of fluid ABCD be perpendicular to the horizon. Take any points, x and y, at different depths, and conceive the column to be divided into a number of equal spaces by horizontal planes. Then, since the density of the fluid is uniform throughout, the pressure upon x and y, respectively, must be in proportion to the number of equal spaces above them, and consequently in proportion to their depths.

Fig. 171.



Case 2. Let the column be of the same perpendicular height as before, but inclined as is Fig. 171, (2); then its quantity, and of course its weight, is *increased* in the same ratio as its length exceeds its height; but since the column is partly supported by the plane, like any other heavy body, the force of gravity acting upon it is *diminished* on this account in the same ratio as its length exceeds its height; therefore as much as the pressure on the base would be augmented by the increased length of the column, just so much it is lessened by the action of the inclined plane; and the pressure on any part of C*c* will be, as before, proportioned to its perpendicular depth; and the pressure of the inclined column AC*ac* will be the same as that of the perpendicular column ABCD.

There are various experiments also by which this proposition is fully established.*

386. According to Art. 379, the lateral is equal to the downward pressure; and consequently on this principle, may easily be estimated the amount of pressure on the sides of any column of water, or on the banks of rivers, canals, &c. At the depth of 8 feet, the pressure on a square foot is equal to the weight of a column of water, whose base is 1 foot and depth 8 feet, and consequently its solid contents 8 cubic feet; and since 1 cubic foot of water weighs 1000 ounces or $62\frac{1}{2}$ lbs. therefore the weight of the column $=8\times62\frac{1}{2}=$ 500 lbs. Hence the pressure on a square foot, at different depths, will be as in the following table.

Depth in feet.	Pressure on a square foot.	Depth in feet.	Pressure on a square foot.
8	500 lbs.	56	3500 lbs.
16	1000	64	4000
24	1500	72	4500
32	2000	80	5000
40	2500	88	5500
48	3000	96	6000

1 mile or 5280 feet, . . . 330,000 lbs.
5 " 1,650,000

Hence it appears that at the moderate depth of 64 feet, the pressure of a column of water on the bottom or sides of the containing pipe, becomes 4000 lbs. to the square foot; and the pressure on the bottom of the sea, where it is one mile in depth, is 330,000 lbs. to the square foot, and where it is five miles deep, that pressure is no less than 1,650,000 lbs.† From these considerations, we may readily apprehend the cause of the great difficulty experienced in confining a high column of water; and hence also may be inferred the immense pressure that is exerted on the bottom of the sea. It is said that the Greenland whale sometimes descends to the depth of a mile, but always comes up exhausted and blowing out blood, showing that the pressure had so acted upon the vessels as to cause them to discharge a portion of their contents into the lungs.‡

* Experimental illustrations are, in this part of the work, supposed to be given by the instructor.

† Allowance must also be made for the saltness of the sea, salt water being heavier than fresh. ‡ Ed. Phil. Jour. *Jan.* 1832.

387. Indications of this vast pressure in deep waters, are manifested by several interesting facts. It has long been known to mariners, that if a common square bottle be let down into the sea, its sides are crushed inwards before it has reached the depth of ten fathoms. If a stronger bottle, (a common junk bottle, for example,) be filled with water, corked close, and let down to a certain depth, either the cork will be forced inwards, or if that be secured in its place, the salt water will make its way into the bottle in spite of it, either by compressing the cork or by forcing in water through it. It was by sinking an apparatus to the depth of 500 fathoms, that Mr. Perkins first proved the compressibility of water, as mentioned in Art. 377. The apparatus consisted of a hollow brass cylinder, resembling a small cannon,* and furnished with a stopper so contrived as to indicate, when the apparatus was drawn up, how far it had been driven in while at the lowest depth. The same experiments were afterwards repeated on shore, a pressure being applied to the plug, by means of the hydrostatic press, equivalent to 2000 atmospheres.

The increase of pressure in proportion to the depth of the fluid, renders it necessary to make the sides of pipes or masonry, in which fluids are to be contained, stronger the deeper they go. The same remark applies to dams, flood-gates, and banks.

388. At the depth of one mile the compression of water is $\frac{1}{13\frac{1}{4}\frac{1}{7}}$ of its bulk, and its specific gravity is increased in the same ratio; so that bodies which sink near the surface of the sea, may float at a certain depth before they reach the bottom. On the other hand, a porous body, that is light enough to float near the surface, will have so much water forced into its pores, when it is sunk to a great depth, as never to rise. This is the case with ships that are wrecked in deep water; the parts of the wreck do not rise to the surface, as they do in shallow water.†

389. *The pressure of a fluid against any surface, in a direction perpendicular to it, varies as the area of the surface multiplied into the depth of its center of gravity below the surface of the fluid.*

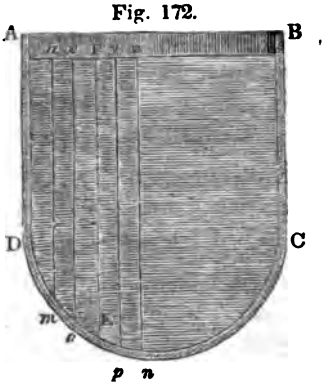
* A cannon itself was afterwards employed in these experiments.

† Arnott.

When a portion, as a square foot, of the lateral surface of a column of water, is taken, all parts of it are not equally distant from the surface of the fluid; and, in this case, the *average* depth, or (which is the same thing) the depth of the center of gravity, is to be understood.

Let *m*, *n*, (Fig. 172.) be a given portion of the vessel ABCD, filled with water or any liquid, and let us conceive this portion to be

Fig. 172.



occupied by any number of particles *m*, *o*, *p*, *n*, &c.; then the pressure produced by all these particles will be (Art. 83.) $m \times mu + o \times ox + p \times py + n \times nz$, &c.: but, by a property of the center of gravity,* the sum of the products is equal to the sum of the particles, that is, the area of the surface, multiplied into the distance of the center of gravity from the surface of the fluid.†

Hence, the pressure on the side of a cubical vessel, filled with fluid, is one half the pressure against the bottom; and the whole pressure against the sides and bottom, is equal to three times the weight of the fluid in the vessel.

390. *Fluids rise to the same level in the opposite arms of a recurved tube.*

Let ABC be a recurved tube: if water be poured into one arm of the tube, it will rise to the same height in the other arm. For, by Art. 385, the pressure upon the lowest part at B, in opposite

* See the last step in the demonstration of Art. 83, from which $GK \times (p + p' + p'') = p \times px + p' \times p'x' + p'' \times p''x''$.

† Ed. Encyc. 'Hydrodynamics.'

directions, is proportioned to its depth below the surface of the fluid. Therefore, these depths must be equal, that is, the height of the two columns must be equal, in order that the fluid at B may be at rest; and unless this part is at rest, the other parts of the column cannot be at rest. Moreover, since the equilibrium depends on nothing else than the *heights of* the respective columns, therefore, the opposite columns may differ to any degree in quantity, shape, or inclination to the horizon. Thus, if vessels and tubes very diverse in shape and capacity, as in Fig. 174, be connected with a reservoir, and water be poured into any one of them, it will rise to the same level in them all.

Fig. 173.

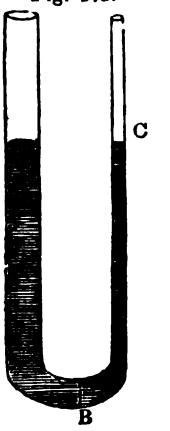
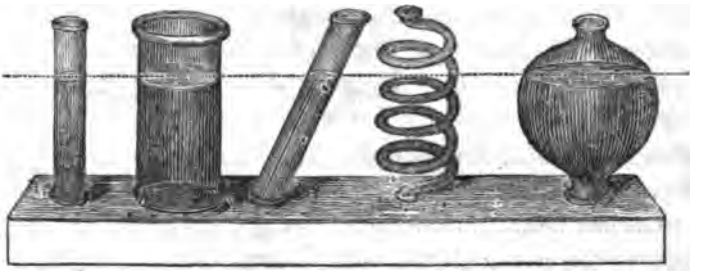


Fig. 174.



The reason of this fact will be farther understood from the application of the principle of *equal momenta*, (Art. 149.); for it will be seen that the velocity of the columns, when in motion, will be as much greater in the smaller than in the larger columns, as the quantity of matter is less; and hence the opposite momenta will be constantly equal.

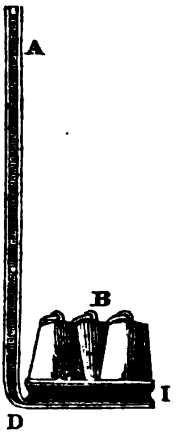
391. Hence, water conveyed in aqueducts or running in natural channels, will rise just as high as its source. Between the place where the water of an aqueduct is delivered and the spring, the ground may rise into hills and descend into valleys, and the pipes which convey the water may follow all the undulations of the country, and the water will run freely, provided no pipe is laid higher than the level of the spring. Waters running in natural channels in the earth are governed by the same law.

392. The aqueducts constructed by the ancient Romans, were among the most costly monuments of their arts. Several of them were from thirty to one hundred miles in length, and consisted of vast covered canals built of stone. They were carried over valleys and level tracts of country upon arcades, which were sometimes of stupendous height and solidity.* From the fact that the ancients built aqueducts with so much labor, raising them to a great height in crossing valleys, instead of availing themselves of the principle under consideration, some have supposed that they were unacquainted with this principle. It appears nevertheless that they were acquainted with it, and even understood the use of pipes in conveying water; but probably the expense of pipes, and the difficulty of making them strong enough to resist the pressure when laid at a considerable depth below the source, prevented their general use.

393. *The pressure upon the horizontal base of any vessel containing a fluid, is equal to the weight of a column of the fluid, found by multiplying the area of the base into the perpendicular height of the column, whatever be the shape of the vessel.*

This follows from Art. 389, since here, the distance of the center of gravity of the base from the surface of the fluid, is the same as the perpendicular height of the column. With a given base and height, therefore, the pressure is the same whether the vessel is larger or smaller above, whether its figure is regular or irregular, whether it rises to the given height in a broad open funnel, or is carried up in a slender tube. Hence, *any quantity of water, however small, may be made to balance any quantity, however great.* This is called the *hydrostatic paradox.* The experiment is usually performed by means of a water bellows, as represented in Fig. 175. When the pipe AD is filled with water, the pressure upon the surface of the bellows, and consequently the force with which it raises the weights laid on it, will be equal to the weight of a cylinder of water, whose base

Fig. 175.



* Bigelow, El. Tech. 303.

is the surface of the bellows, and height that of the column AD. Therefore, by making the tube small, and the bellows large, the power of a given quantity of water, however small, may be increased indefinitely. The pressure of the column of water in this case corresponds to the force applied by the piston in the Hydrostatic Press, (Art. 381.) and the explanation according to the principle of equal momenta, is the same in both cases.*

394. The principle of the Hydrostatic Paradox, is sometimes exemplified by pouring liquids into casks, through long tubes inserted in the bung holes. As soon as the cask is full, and the water rises in the pipe to a certain height, the cask bursts with violence. The same cause is supposed sometimes to produce great effects in nature, such as splitting rocks, heaving up mountains, and other effects resembling earthquakes. For, suppose that in the interior of a mountain there were an empty space ten yards square, and only an inch deep, in which the water had lodged so as to fill it entirely; and suppose that a crevice in the earth should extend from this spot two hundred feet above, which should also become filled with water by rain or otherwise: the force exerted would be adequate to shake the mountain, and perhaps rend it asunder.†

395. Although the weight of a given quantity of water would not be altered by varying the shape of the vessel, yet the pressure which it exerts on the bottom of the vessel will be greater in proportion as the altitude of the mass is greater, and of course greater in a narrow vessel than in a wide one. If it be asked why the weight is not increased as the downward pressure is increased, the answer is, that the pressure in that direction is exactly counterbalanced by an equal pressure in the opposite direction.

OF SPECIFIC GRAVITY.

396. *The Specific Gravity of a body, is its weight compared with the weight of another body of the same bulk, taken as a standard.*

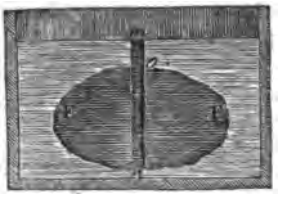
* The bellows rises through so small a space, that its motion is hardly perceptible, but it may be rendered very striking by connecting with the bellows (as is done in the lecture room at Yale College,) a lever, and several multiplying wheels, which give rapid motion to a pointer.

† Library of Useful Knowledge, 'Hydrostatics.'

Water is the standard for all solids and liquids, and common air for gases. Therefore the specific gravity of a solid or a liquid body, is the ratio of its weight to the weight of an equal volume of water; and the specific gravity of an aëriform body, is the ratio of its weight to the weight of an equal volume of air. But a ratio is expressed by a vulgar fraction, whose numerator is the antecedent, and whose denominator is the consequent. If, therefore, the weight of a body is made the numerator, and the weight of an equal volume of water the denominator, the value of the fraction, that is, the quotient, will express the specific gravity of the body. Hence, the weight of a body being given, and being made the numerator, every process for finding the specific gravity, consists in finding for the denominator the weight of an equal bulk of water or air. The principles upon which the methods of doing this depend, are now to be explained.

397. *A body immersed in a fluid, loses as much weight as is equal to the weight of an equal volume of the fluid.*

Fig. 176.



Let EF be a solid body immersed in a vessel of water or any fluid, and suppose it divided into an indefinite number of perpendicular columns, reaching to the surface of the fluid, as *mon*. Now the upward pressure at *n* is as its depth, and the downward pressure at *o* as its depth; therefore the upward pressure exceeds the downward, by the weight of a column of water equal to *no*. The same is true of all the columns, however numerous they may be, that can be drawn parallel to *no*; but these columns, taken collectively, make up a body of water equal in bulk to the solid. Hence the solid is pressed upwards more than downwards, by the weight of a quantity of water of the same magnitude, and consequently loses so much of its weight. Hence the specific gravity of any solid body that will sink in water, is found by the following

RULE.—*Divide the weight of the body by its loss of weight in water.*

398. When the body whose specific gravity is required is lighter than water, as a cork, for example, the object is still to find the weight of an equal bulk of water, since that will constitute the denominator, or divisor, as before. To ascertain this, suspend any heavy body, as a mass of lead or glass, in water, and find its weight. Attach to it the lighter body. Now the cork will not only lose all its own weight, but will diminish the weight of the heavy body; and the weight of an equal bulk of water will be indicated by the whole of what the cork loses, namely, its own weight added to the loss occasioned to the other body. Whence we have the following

RULE.—To find the specific gravity of a body lighter than water, *Divide its weight by the sum of its weight added to the loss of weight which it occasions in a heavy body previously balanced in water.*

399. A solid which is soluble in water, as a lump of salt, is protected from solution by smearing it with oil or a thin coat of bees' wax; and solids that are very porous and would absorb water, and thus increase their specific gravities, as certain kinds of wood, are first covered with varnish.* The specific gravity of solid substances, which are too minutely divided to be weighed in water, separately, as grains of sand or shot, may be found by weighing them in a small bucket previously balanced in water.

400. The specific gravity of *liquids* may be ascertained by several different methods.

RULE I.—*Weigh equal volumes of the liquid and of water, and divide the former result by the latter.*

RULE II.—*Ascertain the loss of weight of any solid body, first in the liquid and then in water, and divide the former result by the latter.*

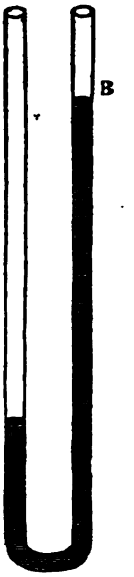
Both these rules obviously depend upon the same principles as those explained in Art. 396, the weight of the liquid being immediately compared with that of an equal bulk of water; but there is another method, founded on the following proposition.

* Cavallo, I. 212.

401. *Two columns of fluids of different specific gravities, pressing freely on each other at their bases, balance one another, when their heights are inversely as their specific gravities.*

Let AB be a recurved tube, and let the height of the column of the fluid B be as much greater than that of A, as the fluid B is lighter than the fluid A; the two columns will then be in equilibrio.

Fig. 177.



If the tube be of uniform bore throughout, then the proposition is manifestly true, because the quantities of matter pressing on each other in opposite directions will be equal and will have equal momenta; but from the peculiar nature of fluids, (Art. 390.) the opposite pressures will be the same, when the heights of the columns are the same, whatever may be the shape or capacity of the tube. If we introduce mercury into one arm of the tube and water into the other, the graduated scale will indicate that the water stands 13½ times as high as the mercury. Therefore the specific gravity of mercury is 13½. Proof spirit will stand at .923; sweet oil at .915; and their specific gravities are the same, water being 1.

402. These comprehend the most common methods of determining the specific gravities of bodies; but when great accuracy is required, the process of finding the specific gravity becomes one of much delicacy. The balance must be very sensible; the body when suspended in water must hang by a fine thread or hair, and the water must be of a standard temperature, since it alters its density by a change of temperature, so that the quantity of water which we wish to find, (namely, a quantity equal in magnitude to the given body,) will weigh more when the water is cold and less when it is warm.*

As expeditious methods of finding the specific gravity are sought for in commerce and the arts, various *instruments* have been invent-

* The standard temperature is not universally agreed on. The temperature of 60° is usually understood, at which a cubic foot of water weighs 62.353 lbs. or a little less than 1000 ounces. A better standard is that of 40°, at which temperature the weight of a cubic foot is still nearer to 62½ lbs. or 1000 ounces.

ed for this purpose, which are called in general, *hydrometers*. But before we can understand the principles on which these instruments depend, we must attend to the theory of floating bodies, so far as it is contained in the following propositions.

403. *If a body floats on a fluid, it displaces as much of the fluid as is equal to its own weight.*

For the body P+Q* is supported by the pressure of the fluid upwards against the part immersed. But before the body was immersed, the same pressure supported a quantity of the fluid which occupied the same space and was therefore of the same magnitude with Q; and since the same pressure must sustain the same weight, when there is an equilibrium, the weight of the body must be equal to the weight of a quantity of the fluid equal in bulk to Q.

This proposition is also easily established by the experiment; for if into a vessel full of water a floating body, as a piece of wood, be introduced, the quantity of water displaced will be found to be exactly equal in weight to the body. Or if the vessel full of water be accurately balanced in a scale, and then removed and the piece of wood introduced, on restoring the vessel it will still remain in equilibrium, the wood exactly compensating for the water it displaced.

404. *If a body floats on a fluid, the part immersed bears the same ratio to the whole body, as the specific gravity of the body bears to that of the fluid.*

Let S be the specific gravity of the fluid, and s that of the solid; then the absolute weight of the body (P+Q)* will equal (P+Q) $\times s$,† and the weight of a quantity of the fluid equal to Q is Q$\times$S; and by Art. 403, these weights are equal to each other, that is,

$$(P+Q)\times s = Q\times S \;\therefore\; Q : P+Q :: s : S.$$

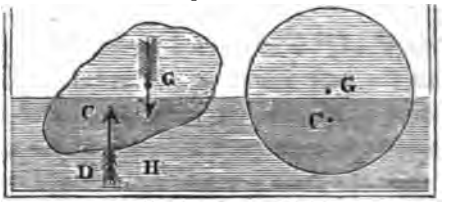
405. *A floating body will be at rest only when its center of gravity is in the same vertical line with the center of gravity of the part of the fluid displaced.*

* Q represents the part immersed; P the other part.

† s denotes the specific gravity of the body, water being 1; and $s = \frac{W}{P+Q}$ (Art. 396.) $\therefore W = (P+Q)\times s$.

As the space from which the fluid has been displaced, exactly copies the figure of the part of the body immersed, the center of gravity of the displaced fluid is the same with that belonging to the figure of the segment. Let this center of gravity be at C, (Fig. 178.) while that of the entire body is at G. Now the fluid, pre-

Fig. 178.



vious to its removal, was sustained by an upward force equal to its own weight, acting in the vertical line DC; and the same upward force now acts upon the floating body at the point C. But the body, being free to move, is carried downwards by a force acting in the direction of the vertical line GH. Were these two forces exactly opposite and equal, they would keep the body at rest; but this is the case only when the points C and G are in the same vertical line: in every other position of these points the two parallel forces tend to turn the body round. (Art. 70.)

When the floating body is a regular solid, as a sphere, the space occupied by the immersed segment being always the same, its center of gravity is immovable. If the body is of uniform density, so that the center of gravity coincides with the center of magnitude, it will remain at rest in every position; for the center of gravity of the whole body and of the segment, will always be in the same vertical diameter. But if the sphere is of unequal density, then it will turn around the center of gravity of the segment as around a fixed point, and be at rest, only when its center of gravity is either directly above or directly below that of the segment.

Persons have sometimes attempted *to walk on the water*, by attaching to the feet inflated bladders, or air bags of India rubber. The body can be kept from reversing its position, only by maintaining its center of gravity directly over that of the fluid displaced,—a point of such delicacy, as to require great skill and dexterity. The feat is rendered easier by carrying a staff having a blown bladder at

the end of it, by which three points of support are commanded, and consequently a greater breadth of base is secured. Most persons, however, in attempting this feat, would shortly find their heads downwards, and their feet out of water.

Life preservers, consisting of air bags attached to the waist, act on the foregoing principles, and, in numerous instances, have been instrumental in saving the lives of shipwrecked mariners.

406. The HYDROMETER, an instrument used in commerce for determining at once the specific gravity of liquors, depends on the principle enumerated in Art. 404.

The most common Hydrometer is represented in Fig. 179. It consists of a hollow ball, with a long slender stem, and since it will sink until it has displaced a quantity of the fluid equal in weight to itself, it will of course sink to a greater depth the lighter the fluid.

Fig. 179.

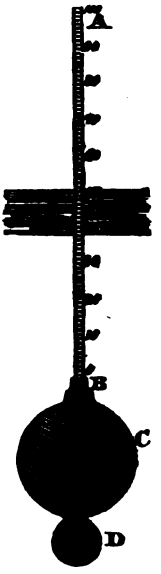
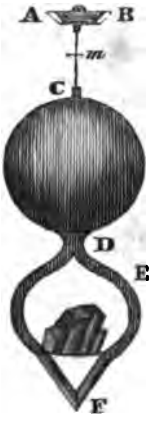


Fig. 180.



From the depths to which it sinks, therefore, as indicated by the graduated stem, the corresponding specific gravities are estimated according to the formula in Art. 404, and arranged in a table. *Nicholson's Hydrometer*, (Fig. 180.) is the most useful of this class of instruments, since it may be applied to finding the specific gravities of solid bodies.

In addition to the hollow ball of the common hydrometer, it is furnished with a dish AB for receiving weights, and a stirrup EF for holding the substance under trial. The instrument is so adjusted that when 1000 grains are placed in the upper dish, the whole will sink in distilled water at the temperature of 60° Fahr. to the point m in the middle of the stem. If therefore in the case of different *fluids*, we add weights until m stands at the level of the fluids, the quantities displaced will be the same in all, and the weight of each quantity will be that of the instrument W together with the weight w added to the dish, which will constitute the *numerator* of the fraction expressing the specific gravity, (Art. 396.) while $W+1000$ expresses the *denominator*, being the weight of an equal bulk of water; and since the weight of the instrument is known, the only thing to be determined in any case is the weight to be added, in order to sink the point m to the level of the fluid. Then $\frac{W+w}{W+1000}$ = the specific gravity.

To determine the specific gravity of a *solid* body not exceeding 100 grains in weight, we first place the instrument in water, put the body into the dish, and add weights to sink the point m to the level of the water. Then, since the weight of the body itself and the weight added, together equal 1000 grains, the weight of the body $=1000$ – the weight added. Now we transfer the body to the stirrup EF, and ascertain what weight must be added to the dish to sink it to the same level as before. This weight will show the loss of weight of the body in water; and the weight in air, divided by the loss of weight in water, gives the specific gravity. As the cylindrical stem of the instrument is only one fortieth of an inch in diameter, the instrument will rise or fall nearly one inch by the subtraction or addition of one tenth of a grain. It will therefore indicate a change of weight less than one twentieth of a grain, and give the specific gravity correct to five places of decimals.*

407. An accurate knowledge of the specific gravities of bodies, is of great use for many purposes of science and the arts, and they have therefore been determined with the greatest possible precision. The heaviest of all known substances is platina, whose specific gravity, in its state of greatest condensation, is 22,† water being 1; and

* Ed. Encyc. X. 781. † Ed. Encyc. X. 794.

the lightest of all ponderable bodies is hydrogen gas, whose specific gravity is .073, common air being 1. Also, air is 818 times lighter than water. Hence, by calculation, it will be found that platina is about 247,000 times as heavy as hydrogen, and a wide range is allowed to the various bodies which lie between these extremes. The metals, as a class, are the heaviest bodies; next to these come the metallic ores; then the precious gems; and finally, minerals in general, animal, liquid and vegetable substances, in order, according to the following table.

Metals, (pure,) not including the bases of the alkalies and earths, from - - - - - - - - 5 to 22

Gold,	19.25	Steel,	7.84
Quicksilver,	13.58	Iron,	7.78
Lead,	11.35	Tin,	7.29
Silver,	10.47	Zinc,	7.00
Copper,	8.90		

Metallic Ores, lighter than the pure metals, but usually above 4.00*

Precious Gems, as the ruby, sapphire, and diamond, 3—4

Minerals, comprehending most stony bodies, - - 2—3

Liquids, from ether highly rectified to sulphuric acid highly concentrated, - - - - - - - - ¾—2†

Acids, in general, heavier than water.

Oils, do. lighter; but the oils of cloves and cinnamon are heavier than water; the greater part lie between .9 and 1. - - - - - - - - - .9—1

Milk, - - - - - - - - - - 1.032

Alcohol, (perfectly pure,) - - - - - .797

Do. of commerce, - - - - - - .835

Proof Spirit, - - - - - - - - .923

Wines; the specific gravity of the lighter wines, as Champagne and Burgundy, is a little less, and of the heavier wines, as Malaga, a little greater, than that of water.

Woods, cork being the lightest, and lignum vitæ the heaviest, ¼ to 1⅓‡

* In a few instances, metallic ores, when largely combined with foreign ingredients, fall below 3. † Ed. Encyc.

‡ The lightest kinds of wood, are poplar, pear, and sassafras, all of which are below .5; the heaviest, are ebony, cedar, mahogany, oak-heart, box, and pomegranate, which are heavier than water. Wood increases in density by age. Knots have been known to reach the specific gravity of 1.76.

408. If we balance, in a pair of scales, a tumbler filled with water, to a certain mark near the top, and then, turning out all the water except a small quantity, introduce any solid body, (as a tumbler a little less than the first,) so as to raise the water on the sides to the same mark as before, the equilibrium will be restored. Here, the space occupied by the solid immersed, is the same with that before occupied by the water. On the same principle, a ship is floated in a dock with a very small quantity of water, and still rides as freely as on the ocean. By the ascent of the water on the sides, the upward pressure on the bottom is increased, on the same principle as in the Hydrostatic Paradox, (Art. 393.) Though in this case, we cannot say that a quantity of water is *displaced* equal in weight to the solid, (since the whole of the water originally in the vessel may not have been nearly sufficient to fill the space occupied by the ship,) yet the effect is the same, in regard to the pressure on the water below the ship, and of course on the upward pressure, (Art. 379.) as though the space occupied by the ship below the level of the fluid on its sides, were filled with water. On this principle, the weight of a loaded boat in the lock of a canal is easily estimated.

Boats are sometimes made of iron instead of wood, their thickness being so much less, that the entire weight of the boat, is not greater than when made of wood.

409. The human body, when the lungs are filled with air, is lighter than water, and but for the difficulty of keeping the lungs constantly inflated, it would naturally float.* With a moderate degree of skill, therefore, swimming becomes a very easy process, especially in salt water. When, however, a man plunges, as divers sometimes do, to a great depth, the air in the lungs becomes compressed, and the body does not rise except by muscular effort. The bodies of drowned persons rise and float after a few days, in consequence of the inflation occasioned by putrefaction. Quadrupeds swim much more easily than man, because the motions of the limbs necessary to sustain themselves, nearly coincide with their natural motions in walking, while the body maintains nearly its usual posture.

* See Dr. Franklin's remarks on this subject, in his Works.

410. *If a body is held beneath the surface of a fluid, the force with which it will ascend, if it is lighter than the fluid, or with which it will descend, if it is heavier, is equal to the difference between its own weight and the weight of an equal quantity of the fluid.*

Let s be the specific gravity of the solid and S that of the fluid, and M the weight of an equal bulk of water. Now the body held beneath the water obviously descends with its own weight $=M \times s$,* while it is pressed upward with the weight of the quantity of fluid displaced $=M \times S$; consequently, the force with which it ascends must be $(M \times S) - (M \times s)$, and the force with which it descends is $(M \times s) - M \times S$, which are the differences between the weight of the body and the weight of the fluid displaced.†

411. On the foregoing principle, is founded the construction of a machine called the Camel, for raising sunken vessels, or for lifting ships over sand banks. A similar effect is exibited in rivers, where the ice is formed upon the stones at their bottom. Ice is specifically lighter than water, and therefore when it accumulates to a certain degree round the stones, the upward pressure upon the stones exceeds their pressure downwards, and they are brought to the surface, having been sometimes torn up with great force. Huge masses of stones appear in many cases to have been floated by the ice adhering to them, and carried to a great distance from the place of their formation.‡

412. Rocks and stones being only a little more than twice as heavy as water, of course nearly half their weight is sustained while they are immersed in water; and hence the increased weight which is felt when a large stone is lifted from the bed of a river, as soon as it reaches the surface. Large masses of rocks are transported with far greater facility by torrents, on account of their diminished weight. On the same principle, the limbs feel very heavy after lying for some time in a bath. Life boats have a large quantity of cork mixed in their structure; or of air-tight vessels of thin copper or tin plate, so that, even when the boats are filled with water, a considerable part still floats above the surface.§

* For $s = \frac{W}{M}$ $\therefore W = M \times s$. † Ed. Encyc. X. 722. ‡ Ib.
§ Arnott.

When light bodies are attached to heavier for the purpose of making them float, some caution is necessary, since, on account of the tendency of the center of gravity to seek the lowest point, there will be danger of upsetting. Thus, persons endeavoring to swim by attaching to their persons blown bladders, sometimes have their heads turned downwards; and a man undertaking to walk on the water in a pair of cork boots, would shortly disappear and nothing would be seen except the boots sticking out of the water.* (Art. 405.)

413. The *magnitude* of bodies may frequently be most conveniently and accurately estimated from the doctrine of specific gravities. Suppose we wish to ascertain the exact number of solid inches contained in a stone of rude and irregular shape, we should find great difficulty in applying to it any linear measurement; but if we ascertain its loss of weight in water, then we have the weight of an equal bulk of water, and since 1000 ounces contain 1728 cubic inches, we may easily find how many cubic inches correspond to the weight of water of equal magnitude with the body in question.

We may estimate the quantity of ice in an island of ice, such as occur in the northern seas, by applying the principle demonstrated in Art. 404.

414. As examples of the manner in which questions are solved by the principles of Hydrostatics, we subjoin a few problems of the more difficult class, with their solutions, to which we shall add a variety of questions as an exercise for the learner.

PROBLEMS.

1. What is the weight of a chain of pure gold, which raises the water 1 inch in height, in a cubical vessel whose side is 3 inches? and suppose a chain of the same weight were adulterated with 14½ oz. of silver, how much higher would it raise the water in the vessel?

The contents of the chain are $3\times3\times1=9$ cubic inches.

Since a cubic foot or 1728 inches of distilled water weigh 1000 oz.† avoirdupois, and since 192 oz. avoirdupois = 175 troy, in which

* Arnott.

† By act of Parliament, 1825, 19 cubic inches of distilled water, temp. 50° Fahr. weigh 10 oz. troy. These numbers may be used instead of those in the text.

gold and silver are weighed; therefore, 1728 inches of water weigh 911.458 oz. troy; or 1 cubic inch weighs .5274 oz. The specific gravity of gold being 19.25, and that of silver being 10.47, therefore 1 *inch of gold weighs* 10.15 *oz. and* 1 *inch of silver weighs* 5.52 *oz.* Hence the chain weighs 91.35 oz. Again,

10.15 oz. : 1 in. :: 14.5 oz. : 1.428 = cubic inches of gold in 14.5 oz.

5.52 : 1 :: 14.5 : 2.626 = do. silver do.

$\therefore 9 - 1.428 + 2.626 = 10.198$ = column of water raised by the chain;

and $\frac{10.198}{9} = 1.133$ = whole height, and $1.133 - 1 = .133$ Ans.

2. The specific gravity of lead being 11.35; of cork, .24; of fir, .45: How much cork must be added to 60 lbs. of lead, that the united mass may weigh as much as an equal bulk of fir?

Let x = weight of the cork in pounds.

Then $60 + x$ = weight of united mass = weight of an equal bulk of fir.

$\therefore \frac{60+x}{.45}$ = do. of an equal volume of water.

And $\frac{60}{11.35} + \frac{x}{.24}$ = the same, and therefore = $\frac{60+x}{.45}$

Hence, $x = 65.8527$ lbs. Ans.

3. A cone, whose height is 6 inches, is immersed in water, with its vertex downwards. Its specific gravity being $\frac{1}{8}$th that of the water, what part of the axis will be immersed?

Let x = part of the axis immersed:

Then $x^3 : 6^3 :: 1 : 8 . \therefore x = \left(\frac{216}{8}\right)^{\frac{1}{3}} = \frac{6}{2}$ = *one half the axis.* Ans.

4. A sailor had a 10 gallon cask half full of brandy, and not being allowed to keep liquors on board, he threw it overboard for the purpose of concealment, attaching to it a mass of lead just sufficient to keep it under water: Required *the weight of the lead*, the wood of the cask containing 216 cubic inches, the sp. gr. of the brandy being .886, that of sea water 1.030, that of the lead 11.35, and that of the wood .8?

The whole quantity of water displaced = $231 \times 10 + 216 = 2526$ inches; and $231 \times 5 = 1155$ = quantity of brandy.

Then, 1728 : 1030 : : 2526 : 1505.66 = wt. of water.
1728 : 886 : : 1155 : 592.2 = wt. of brandy.
1728 : 800 : : 216 : 100. = wt. of wood.

Then 1505.66 − (592.2 + 100 = 692.2) = 813.46 = weight of the water above that of the brandy and cask, which is the buoyant power to be counteracted by the lead.

Now, how much lead will it take to weigh 813.46 in water?

Let x = wt. of lead required; then $\frac{x}{11.02^*}$ = weight of an equal bulk of water. Therefore $x - \frac{x}{11.02} = 813.46$, and $x = 894.64$ oz. *or nearly* 56 lbs. Ans.

QUESTIONS ON HYDROSTATICS.

1. *In a Hydrostatic Press,* (Fig. 168.) *the height of the small column* AB *on which the power acts is* 2 *feet above the bottom of the large piston* CD; *the diameter of the cylinder* AB *is* 1 *inch, and of the cylinder* CD 1 *foot. By means of a screw turned by a lever, a man can exert a force on* A *equal to* 500 lbs.; *What amount of pressure can he apply with the aid of this press, combining his own strength with the pressure of the column of water* AB?†

Ans. 72098.17 lbs.

2. *A Junk bottle, whose lateral surface contained* 50 *square inches, was let down into the sea* 3000 *feet: What pressure would the sides of the bottle sustain, no allowance being made for the increased specific gravity of the sea water?* Ans. 65104.166 lbs.

3. *A Greenland whale sometimes has a surface of* 3600 *square feet: What pressure would he bear at the depth of* 800 *fathoms?*

Ans. 1080,000,000 lbs. or more than 482142 tons.

4. *A Mill-Dam, running perpendicularly across a river, slopes at an angle of* 25 *degrees with the horizon. The average depth of the stream is* 12 *feet, and its breadth* 500 *yards: Required the amount of pressure on the dam?* Ans. 15971906 lbs. or 7130+tons.

* The specific gravity of lead above sea water.

† It is obvious that the elevation, and consequently the pressure, of this column would be continually diminishing. The question respects only the commencement of the process.

5. *A mineral weighs* 960 *grains in air, and* 739 *grains in water: What is its specific gravity?* Ans. 4.344.

6. *What are the respective weights of two equal cubical masses of gold and cork, each measuring* 2 *feet on its linear edge?*

Ans. The gold weighs 9625 lbs.=4.297 tons; the cork weighs 120 lbs.

7. *On one of the peaks of the Alps, is a single mass of granite rock of a nearly globular shape, which is estimated by measure to contain* 5949 *cubic feet. The whole mass is so nicely balanced on its center of gravity, that a single man may give it a rocking motion. By trial made upon a small fragment, its specific gravity was found to be* 2.6: *What is its weight?* Ans. 431.568 tons.

8. *An island of ice rises* 30 *feet out of water, and its upper surface is a circular plane, containing* $\frac{3}{4}$*ths of a square acre. On the supposition that the mass is cylindrical, required its weight, and depth below the water, the specific gravity of sea water being* 1.0263, *and that of ice* .92? Weight 242900 tons; depth 259.64 feet.

9. *Wishing to ascertain the exact number of cubic inches in a very irregular fragment of stone, I ascertained its loss of weight in water to be* 5.346 *ounces: Required its dimensions?*

Ans. 9.238 cubic inches.

10. *Hiero, King of Syracuse, ordered his jeweller to make him a crown of gold, containing* 68 *ounces. The artist attempted a fraud by substituting a certain portion of silver; which being suspected, the king appointed Archimedes to examine it. Archimedes, putting it into water, found it raised the fluid* 8.2245 *inches; and having found that the inch of gold weighs* 10.36 *ounces, and that of silver* 5.85 *ounces, he discovered what part of the king's gold had been purloined. It is required to repeat the process.* Ans. 28.8 ounces.

CHAPTER II.

OF LIQUIDS OR NON-ELASTIC FLUIDS IN MOTION.

415. That branch of Natural Philosophy which treats of fluids *in motion*, is usually denominated *Hydraulics*, from ὕδωρ, *water*, and αὐλὸς, *a torrent*. It embraces, therefore, the phenomena exhibited by water issuing from orifices in reservoirs—projected obliquely or perpendicularly—flowing in pipes, canals, and rivers—oscillating in waves—or opposing a resistance to the progress of solid bodies.*

In this part of the doctrine of fluids, the deductions of *theory alone* are of little value, and are in fact so discordant with experience, that little reliance can be placed on them, except as modified by experiment. When thus modified, the truths learned respecting the laws that govern the motions of fluids, have a high degree of practical utility.

The causes of this discordance between theory and experiment are such as the following;—the different states in which the same fluid is found in consequence of changes of temperature, and different degrees of purity—the attraction existing between its particles and preventing that perfect fluidity which is contemplated by the definition, and upon which our reasonings are founded—the friction against the sides of the vessel—the resistance of the air—the size of the vessel in proportion to the aperture—the shape of the aperture itself—the different directions in which the various parts or *filaments* (as they are called) of the fluid run towards the aperture—and the vortices, or irregular motions which are communicated to the fluid by a variety of causes, even by an obstacle to the stream at some distance from the aperture.†

416. The manner in which vessels of water discharge themselves through small orifices in the bottom or sides of the vessels, has been carefully observed by introducing into a column of water contained in a glass vessel, small solid particles, which render the currents of

* Ed. Encyc., Art. *Hydrodynamics.* † Cavallo, I. 267.

the fluid visible. From such observations it appears, that the particles of fluid descend in vertical lines, until they arrive within three or four inches of the orifice, when they gradually turn into a direction more or less oblique, and make directly for the orifice. When the surface of the descending column comes very near to the orifice, a funnel-shaped hollow or cavity makes its appearance, and the various particles which rush towards the orifice, converge to a point without the orifice, *at the distance of the semi-diameter of the orifice itself*, where is the point of greatest contraction, called the *vena contracta*.

From the great number of propositions which contain the doctrine of Hydraulics, we shall select those which appear to be most valuable on account of their practical utility, adding such remarks as will serve to show the modifications to which they are subject in practice.

417. *If a fluid runs through any tube, pipe, or canal, and keeps it constantly full, its velocity in any part of its course, will be* INVERSELY AS THE AREA OF THE SECTION *at that part.*

Let A and a be the areas of two cross sections of a tube of unequal bore, and let V and v denote the velocities of the fluid as it flows through A and a respectively. Now the quantity of fluid which passes through any section, must depend upon the area and velocity conjointly; and since the tube is, by the supposition, kept constantly full, the same quantity runs through every section of the tube in any given time. Hence $A \times V = a \times v$ $\therefore A : a :: v : V$; that is, the velocity is inversely as the area of the section.

If follows from the foregoing proposition, that the velocity of a stream increases as either the breadth or the depth decreases. In a tube, the parts near the axis move with greater velocity than near the sides; and in every canal or river, the velocity of the stream is greater in the central parts, or channel, than at the sides, and greater at the surface than at the bottom. The *mean velocity*, therefore, is to be inferred from the combination of at least three distinct measurements. For example, the velocity of a stream was found to be

On the surface of the channel, - -	5	miles per hour.
At the bottom, - - - - - -	3	" "
At the sides, - - - - - - -	3½	" "

Therefore, the mean velocity $=\frac{5+3+3.5}{3}=\frac{11.5}{3}=3.83$ miles per hour.

It is important to avoid all unnecessary expansions as well as contractions, in pipes or canals, since there is always a useless expenditure of force in restoring the velocity which is lost in the wider parts.*

418. To find the *quantity* of water which flows in a river, we may first ascertain the area of a section by taking the depth in various parts of the section, and dividing the sum of all the depths by the number of measurements, which gives us the *mean* depth. This multiplied into the breadth of the stream, gives the area of the section, which multiplied into the average velocity, (Art. 417.) shows the quantity required.

Example.—Wishing to know the quantity of water that discharged itself per minute through a certain strait, I chose a place below the strait where the water flowed at a nearly uniform rate, where, by noting the time occupied by light substances in floating through various parts of the stream, for a given distance, I found the average velocity to be 96 feet per minute. The mean depth of a section was $8\frac{1}{2}$ feet, and the breadth, 560 feet. Hence,

$$8\tfrac{1}{2}\times 560\times 96=456960 \text{ cubic feet per minute.}$$

419. The phenomena of Rivers have sometimes been explained on the supposition that rivers are bodies falling freely down inclined planes. But the conclusions deduced from this doctrine, are so at variance with experience, as to be of no value.† Were every part of the bed of a river uniform, like a tube, the channel or portion which moves with the greatest velocity, would be in the center of the surface; but inequalities in the sides and bottom usually throw it out of the center, and incline it to one side or the other. The increased velocity of a stream during a freshet, while the stream is confined within its banks, exhibits something of the acceleration which belongs to bodies falling freely down an inclined plane. It

* Young's Lectures on Nat. Phil. I. 283.

† See Robison's Mech. Phil. or Encyc. Brit., Art. *Theory of Rivers.*

presents the case of a river flowing upon the top of another river, and consequently meeting with much less resistance than when it runs upon the rough unequal surface of the earth itself. The augmented force of a stream in a freshet, arises from the simultaneous increase of the quantity of water and the velocity. In consequence of the friction of the banks and beds of rivers, and the numerous obstacles they meet with in their winding course, their progress is very slow, whereas, were it not for these impediments, it would become immensely great, and its effects would be exceedingly disastrous. A very slight declivity is sufficient for giving the running motion to water. Three inches per mile, in a smooth, straight channel, gives a velocity of about three miles per hour. The Ganges, which gathers the waters of the Himalaya Mountains, the loftiest in the world, at the distance of eighteen hundred miles from its mouth, is only eight hundred feet above the level of the sea,—that is, about twice the height of St. Paul's church in London; and to fall these eight hundred feet, in its long course, the water requires more than a month. The great river Magdalena, in South America, running for a thousand miles between two ridges of the Andes, falls only five hundred feet in all that distance.*

The motion of rivers soon becomes uniform, even in very great inclinations, the sum of the resistances forming an equilibrium with the acceleration produced by the descent on an inclined plane. The smallest inclination capable of giving motion to water is a descent of one foot in a million; or about $\frac{1}{12}$th of an inch per mile. A fall of 3 feet per mile, makes a mountain torrent.†

420. *The velocity with which a fluid issues from a small orifice in the bottom or side of a vessel, kept constantly full, varies as the* SQUARE ROOT OF THE DEPTH *below the surface of the fluid.*

The *pressure* at different depths varies as the depth, (Art. 385); also the *momentum* is as the force impressed, and likewise therefore as the depth. Hence, let Q, Q′, denote the quantities of fluid discharged at each orifice respectively; V, V′, the corresponding velocities, and AB, AC, the depths. Then, since the momentum is as $Q \times V$,

$$Q \times V : Q' \times V' :: AB : AC.$$

* Arnott's El. Phys. I. 260. † Rennie, *Reports of Brit. Assoc.* 1834.

But, through a given orifice, the quantity of fluid discharged evidently varies as the velocity, or $Q \propto V$; hence,

$$V^2 : V'^2 :: AB : AC, \text{ or}$$
$$V : V' :: \sqrt{AB} : \sqrt{AC}.$$

It appears therefore, that the fluid issues with the velocity which a body would acquire by falling freely from the surface of the fluid to the orifice, for that is also as the square root of the space. (Art. 29.) And since a body when projected upwards with a certain velocity, will rise to the same height as that from which it must have fallen to acquire that velocity, consequently, if a fluid issue from the side of a vessel through a spout bent upwards, it would, were it not for the resistance of the air, and friction at the orifice, rise to the level of the fluid in the reservoir.

It follows from the foregoing proposition, that an orifice sixteen inches from the surface will discharge twice as much fluid in a given time as one at the depth of four inches; and this is conformable to the law that the pressure varies as the depth; for since twice the quantity flows with twice the velocity, of course the pressure or momentum is four times as great at the depth of sixteen, as at that of four inches. In order, therefore, to draw off from a cistern four times as much water as before, we must place the gate sixteen times as deep. A gate opened in a reservoir at the depth of sixty four inches, will discharge only four times as much as at the depth of four inches.

421. In the construction of water works it is customary to conduct the stream, or such a part of it as is required, into a cubical cistern, and to let it issue from the side of this, near to the bottom, and thus fall upon the main wheel. Instead of admitting the water to the wheel in this manner, it has sometimes been supposed that an advantage might be gained by letting the water fall down a height equal to that of the top of the cistern, perpendicularly upon the wheel, on the supposition that we might thus avail ourselves of the force acquired by the water in falling. But according to the preceding proposition, the force would be the same whether the water issued from the cistern and thus applied itself to the wheel, or whether it fell upon the wheel from a height equal to that of the surface of the water in the reservoir above the orifice. This is true in *theory*;

but in *practice* it would be found more advantageous to take the water out of the cistern, since the force of water falling through the air is considerably diminished by the resistance of the air.

422. *If a cylindrical or prismatic vessel, of which the horizontal section is every where the same, is filled with fluid, and empties itself by an orifice, the velocity with which the surface descends, and also the velocity with which the water issues, is uniformly retarded.*

The velocity with which the surface descends is proportional to that with which the fluid issues from the orifice, and therefore is as the square root of the depth. (Art. 420.) But the velocities of bodies projected perpendicularly upwards are in the same ratio to their spaces, (Art. 30.) and therefore a body projected perpendicularly upwards is in the same relative circumstances as the descending surface of the fluid; and as the projected body is uniformly retarded, the same is true of the descending surface.

On this principle is constructed the *Clepsydra*, or water-clock. Since the descent of the surface is uniformly retarded, the spaces which it describes in equal times, reckoning from the bottom, are as the odd numbers, 1, 3, 5, 7, &c.; and if a cylindrical vessel of water be furnished with an orifice at the bottom which will exactly discharge the whole column in twelve hours, and the sides of the vessel be divided into spaces corresponding to the foregoing numbers, the successive heights of the column become measures of time.

423. *If we accurately mark the time in which a cylindrical or prismatic vessel, whose horizontal section is every where the same, discharges itself to the level of a given orifice, and then draw off for the same time keeping the vessel constantly full, we shall obtain double the quantity of fluid in the latter case as in the former.*

When the vessel is kept constantly full, the velocity at the orifice (and of course the quantity discharged) continues uniformly the same as at first; and since the circumstances of this case are exactly analogous to those of a body projected perpendicularly upwards; and since if a body thus projected were to continue to ascend with the first velocity, it would pass over a space twice as great in the same time as when uniformly retarded; therefore, the truth of the proposition is manifest.

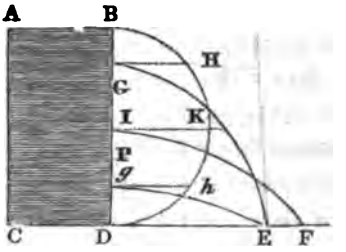
424. A fluid spouting from the side of a vessel, describes the curve of a parabola.

The fluid is precisely in the same circumstances as a projectile acted on by the force of projection (viz. the pressure of the incumbent fluid) and by the force of gravity. Therefore, according to Art. 48, it describes the curve of a parabola. As in the case of other projectiles, the proposition holds good, whatever may be the angle of elevation of the jet.

425. If a semi-circle be described on the perpendicular side of a vessel, as a diameter, and a fluid spout horizontally from any point in the diameter, its random will equal twice the length of the ordinate to that point.

The velocity with which the fluid issues from the vessel, being that which is due to the height BG, (Fig. 181.) is such as if preserved would carry the jet through a space equal to 2BG, in the time of

Fig. 181.



the fall through BG; but after leaving the orifice, it arrives at the horizontal plane in the same time as that in which a body would fall freely through GD. And since, in falling bodies, the times are as the square roots of the spaces, (Art. 29.) therefore, $\sqrt{BG} : \sqrt{GD} :: tBG : tGD$,* that is, the time employed in reaching the plane.† But in the time of describing BG, the jet would be carried uniformly and horizontally over a space equal to 2BG; therefore, to find how far it will proceed in a horizontal direction while it is descending through GD, we have

* t signifies the time of falling down the spaces BG and GD.

† For the jet will reach the plane in the curve, acted on by the two forces of projection and gravity, in the same time in which it would descend through the perpendicular height, urged by gravity alone.

$\sqrt{BG} : \sqrt{GD} :: 2BG : DE = \frac{2BG \times \sqrt{GD}}{\sqrt{BG}} = 2\sqrt{(BG \times GD)} =$ 2GH, that is, double the ordinate to the point G.

The greatest random will be when the fluid spouts from the center, for then the ordinate (and of course twice the ordinate) is greatest; and the randoms will be equal at equal distances above and below the center, for at these points the ordinates are equal.

426. The term FRICTION is applied to the obstruction occasioned to the passage of fluids in the same manner as it is to solids; and it exists to such an extent as to become an object of considerable inconvenience in practice. It can be obviated only by making the conveying pipe of much larger dimensions than would otherwise be necessary, so as to allow the free passage of a sufficient quantity of fluid through the center of the pipe, while a ring or hollow cylinder of water is considered to be at rest all around it. Other circumstances beside friction likewise tend to diminish the quantity of fluid which would otherwise pass through pipes,—such as the existence of sharp or right-angled turns in them, permitting eddies or currents to be formed, or not providing for the eddies or currents that form naturally, by suiting the shape of the pipe to them. It follows, therefore, that whenever a bend or turn is necessary in a water-pipe, it should be made in as gradual a curve or sweep as possible; that the pipe should not only be sufficiently capacious to afford the necessary supply, but should be of a uniform bore throughout, and free from all projections or irregularities against which water can strike, and form eddies or reverberations, since these will impede the progress of the fluid as effectually as the most solid obstacles.

427. An unexpected facility is gained in the discharge of a fluid from the bottom or side of a vessel, by applying a *pipe to the orifice.* On account of the friction known to occur in the passage of a fluid through a tube, it might be supposed that a simple orifice made in the vessel might be more favorable to the discharge of the fluid than an opening prolonged by a tube; but it has been found by experiment, that a vessel of tin, with a smooth hole formed in its bottom, did not discharge water as rapidly as another containing the same weight of water, and an orifice of the same dimensions, to which a short pipe

was applied. By varying the length of the pipe, it is found that when its length is twice its diameter, it produces the most rapid discharge, delivering in this case 82 quarts of water in 100 seconds, while the simple hole delivered but 62 quarts in the same time. If, however, the pipe projects into the vessel, the quantity discharged is diminished instead of being increased by the pipe.*

423. When water is conveyed through a straight cylindrical pipe of any length, the discharge of water may be increased by only altering the *shape of the terminations* of that pipe, viz. by making the end of the pipe which is close to the reservoir, or the entrance to it, of the conical shape of the *vena contracta*, (Art. 416,) and by making the other extremity of the pipe, where the water issues, of a trumpet shape.† By this means, the quantity of water which is discharged in a given time, is more than doubled.‡

WATER WHEELS.

429. Three kinds of water wheels are employed under different circumstances, namely, the *overshot*, the *undershot*, and the *breast wheel*.

The *overshot* wheel is used when the supply of water is scanty, since this construction admits of a more economical use of the water than either of the others. Figure 182, represents a section of an overshot wheel at right angles to the axis. Its diameter is usually nearly equal to the whole fall of the water. It is placed under the head of water in such a way as to receive its whole force into buckets, connected with the rim of the wheel. These buckets are made of such a shape as to retain as much of the water as possible until they reach the lowest point of the wheel, but none at all after passing that point. By this means the weight of the water in the buckets is made to exert as much weight as possible on one side of the wheel, thus causing it to descend, while they oppose little resistance to the

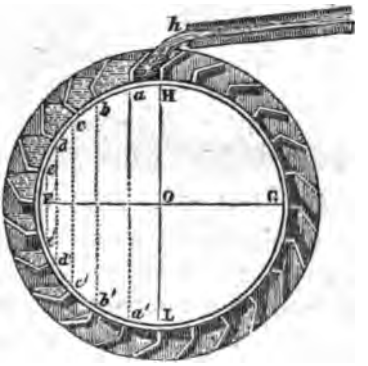
* Ed. Encyc. Art. *Hydrodynamics*.—Millington's Epitome of Nat. Phil. 181.

† Experience shows that the divergency of this termination must not exceed a certain degree, for in that case it will prove rather disadvantageous than useful. It appears that when the divergency is greater than an angle of 16 degrees, the effect ceases entirely; and that the greatest quantity of fluid is discharged, when the divergency is equal to an angle of about 3 degrees.—*Cavallo*.

‡ Cavallo, I. 276.

ascent of the opposite side of the wheel. Let us trace the effect of a single bucket in its revolution. Were it to receive the water directly on the top at H, the only effect would be to cause an increased pressure on the axis of the wheel, while it would not con-

Fig. 182.

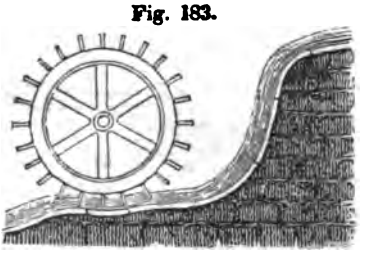


tribute to turn the wheel; and, indeed, within a certain distance from H towards *a*, the weight of the water increases the resistance from friction on the axle more than its force tends to turn the wheel. It is evident, therefore, that the water must begin to fall on the wheel so far towards the side, that its leverage, measured from O on the line OF, may enable it to overcome the friction and all other impediments. As the wheels revolve, the weight of the water acts with a constantly increasing leverage, until at F it acts with its greatest force. From this point, the force declines from two causes, namely, the loss of water from the buckets as their position is gradually reversed, and the diminution of the leverage or effective distance from O on the line OF, until, before it reaches the lowest point L, it may again slightly act as an impediment by increasing, from its weight, the friction on the axle more than it contributes to turning the wheel.

430. There is a certain velocity with which an overshot wheel should move in order to produce the greatest effect. If, on the one hand, the wheel is loaded so heavily that the weight of water is insufficient to move it, then of course the effect is nothing; and if, on the other hand, the velocity of the wheel were to equal that with which water would fall freely, then its pressure on the buckets would

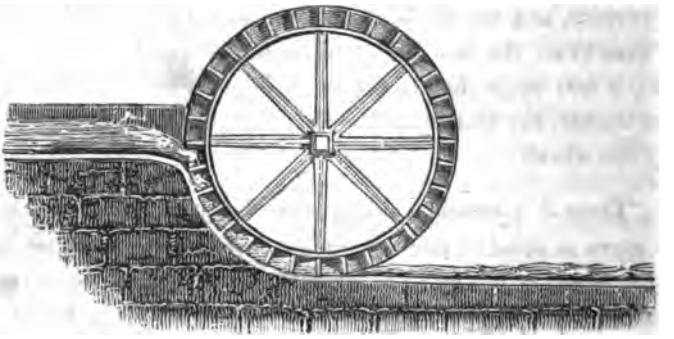
become nothing and its moving power nothing. The best velocity that can be given to an overshot wheel is found to be three feet per second.

Fig. 183.



431. The *undershot wheel* (Fig. 183.) is carried, not by the weight of the water simply, (as is the case in the overshot wheel,) but by the momentum or force of running water. Instead of close buckets for holding water, it is furnished with *float boards*, which receive the impulse of the stream. When these are placed, as in the figure, with their planes at right angles to the rim of the wheel, the latter may turn either way; and this, therefore, is the form of wheels employed in tide mills. When the wheel is required to turn only in one direction, an advantage is gained by placing the float boards so as to present an acute angle towards the current, by which means, the water acts partly by its weight, as in the overshot wheel. The undershot wheel is adapted to situations where the supply of water is always abundant. It acts, moreover, with the greatest effect, when its velocity is half that of the stream.

Fig. 184.



432. The *breast wheel* (Fig. 184.) combines the advantages of both the others, and is therefore adapted to situations where the

supply of water is generally sufficient, but not always abundant. The planes of the float boards are at right angles to the rim of the wheel, and are brought so near to the mill course that the float boards hold water like buckets.

According to Smeaton, the effect of overshot wheels, under the same circumstances of quantity and fall, is, at a medium, double that of an undershot wheel.

CHAPTER III.

OF CAPILLARY ATTRACTION, RESISTANCE OF FLUIDS, AND WAVES.

433. The definition of a fluid, (Art. 376.) proceeds on the supposition that fluids are destitute of cohesion, and that their particles move among themselves without the slightest impediment. All *liquids*, however, have in fact more or less cohesion or mutual attraction among their particles. This is apparent in their forming drops, and in the viscidity of certain liquids, as oil and tar, which on account of this property are sometimes denominated *semi-fluids*. It is owing to this property that water so readily forms itself into drops, and that its surface when viewed in a small cup or wine glass, appears convex. Both of these properties are still more observable in quicksilver, which when poured on a table, forms numerous globules of a perfectly spherical figure; and the convex figure of the surface, as seen in a wine glass, is very striking. When we dip a glass tube into water, it comes out covered with drops of the fluid, which are held by the attraction of the glass for water; but the tube when dipped into quicksilver comes out dry, because the cohesion between the particles of quicksilver for one another is greater, than the mutual attraction that exists between the metal and the glass. Hence, a solid body when immersed in a fluid, is sometimes wet by it and sometimes not, according as the attraction between the solid and the fluid is greater or less than that which exists between the particles of the fluid for one another.

434. If a disk or thin plate of almost any solid substance, as of glass or metal, be suspended from the arm of a balance and counterpoised, upon bringing it into contact with the surface of a fluid capable of wetting it, it will adhere with a considerable force, the amount of which will be indicated by the weights required to be added to the opposite scale, in order to detach it. This experiment shows that a lamina of water is held to a contiguous lamina by a strong force, and hence when a cause operates upon the surface of a fluid to draw up the lamina, a column of fluid may rise along with it, in consequence of the mutual cohesion of the successive laminæ. We see the same principle strikingly exemplified in viscid fluids, as tar, where, on drawing up a small portion of the surface, a column of the fluid follows it. The foregoing fact will lead us to an understanding of the causes of capillary attraction.

435. Capillary Attraction *is the attraction which causes the ascent of fluids in small tubes.*

The tubes must be less than one tenth of an inch in diameter, and tubes whose bores are no larger than a hair, (*capillus,*) present the phenomenon the most strikingly. But though the rise of water above its natural level is most manifest in small tubes, it appears, in a degree, in all vessels whatsoever, by a ring of water formed around the sides with a concavity upwards.*

436. The following are the leading *facts* respecting capillary attraction.

1. *When small tubes, open at both ends, are immersed perpendicularly in any liquid, the liquid rises in them to a height which is inversely as the diameter of the bore.* Though tubes of glass are usually employed in experiments on this subject, yet tubes made of any other material, exhibit the same property. Nor does the thickness of the solid part of the tube, or its quantity of matter, make the least difference, the effect depending solely on the attraction of the surface, and consequently extending only to a very small distance.

2. *Different fluids are raised to unequal heights by the same tube.* Thus, according to Gay Lussac, a tube which will raise water 28 inches will raise alcohol only 9 inches.

* Playfair, *Outlines*, I. 177.

3. A tube $\frac{1}{10}$ of an inch in diameter raises water 5.3 inches; and, since the height is reciprocally as the diameter, *the product of the diameter into the height is a constant quantity*, namely, the .053th part of an inch square.*

4. *Fluids rise in a similar manner between plates of glass, metal, &c.* placed perpendicularly in the fluids and near to one another. If the plates are parallel, the height to which a fluid will rise is *inversely as the distance between the plates;* and the whole ascent is just *half that which takes place in a tube* of the same diameter. If the plates be placed edge to edge, so as to form an angle, and they be immersed in water, with the line of their intersection vertical, the water will ascend between them in a *curve* having its vertex at the angle of intersection. This curve is found to have the properties of the *hyperbola.*

5. If a capillary glass tube be immersed in mercury, *the mercury instead of rising, sinks to a lower level within than without, and its surface is convex instead of concave.*

6. *Tubes which are wider at bottom than at top, elevate fluids to the same height as though the bore were throughout only equal to that of the smaller part.* As this experiment does not succeed *in vacuo*, when the wider end is immersed, the column, in this case, is supported by the pressure of the atmosphere.†

437. Such are the leading facts ascertained respecting capillary attraction. Various explanations of them have been attempted, but that of La Place is most generally received. According to this high authority, the action of the sides of the tube draws up the film of fluid nearest to it, and that film draws along with it the film immediately below it, and so each film drags along with it the next below, until the weight of the volume of fluid raised exactly balances all the forces which act upon it. The fact that the elevation of the water between the parallel plates, is exactly *half* that in a tube of the same diameter, clearly indicates that the force resides in the surrounding body; and the additional fact that the thickness or quantity of that body makes no difference, proves that the force resides in the surface, and that the action extends only to a very small distance. The

* Playfair, *Outlines*, I. 177. † Biot.

concave surface exhibited by water and all fluids capable of wetting the tube (where, of course, the attraction between the fluid and the tube, is greater than between the particles of the fluid among themselves,) still farther indicates a force acting in the direction of the surface of the tube, while the convex surface and depression of mercury, are such effects as might be anticipated from the cohesion of its parts, which greatly exceeds its attraction for glass and other substances.

438. Various *Phenomena* in nature and art are explained upon the principles of capillary attraction. Capillary action is not confined to tubes, but is exerted among all substances which are perforated by pores, or subdivided by fissures or interstices. On this power depend chiefly the functions of the excretory vascular system in plants and animals, and hence also the ascent of humidity through the shivered fragments of rocks, unglazed pottery, gravel, earth and sand. Thus if the pores of the human skin (which are known to be exceedingly small) are estimated at the $\frac{3}{1000}$ part of an inch in diameter, they will support the fluids that circulate through them to the height of 120 inches, or ten feet, or higher than is required for the animal system.* The ascent of the sap in trees has usually been ascribed to capillary attraction, their circulating vessels being a congeries of small tubes; but physiologists now maintain that this action is dependent not on the mechanical structure, but upon something which they denominate the *living principle* of vegetables.

439. According to Professor Leslie, if a soil of gravel contains pores 100th part of an inch in diameter, water will ascend in it by capillary action more than four inches; and supposing the particles of coarse sand to have interstices of the 500th part of an inch, the water would rise through a bed of sixteen inches; and if the pores were diminished to the 10,000 part of an inch, water would rise twenty five and a half feet. Hence, in agriculture, are derived the advantages of deep and perfect tillage; since, the more effectually a soil is pulverized, the better fitted it is to raise and retain water near the surface.

* Leslie, Elem. Nat. Phil. Vol. I.

440. Several familiar examples of capillary attraction may be added. A piece of sponge, or a lump of sugar, touching water by its lowest corner, soon becomes moistened throughout. The wick of a lamp lifts the oil to supply the flame, to the height of several inches. A capillary glass tube, bent in the form of a syphon, and having its shorter end inserted into a vessel of water, will fill itself and deliver over the water in drops. A lock of thread or of candle wick, inserted in a vessel of water in a similar manner, with one end hanging over the vessel, will exhibit the same result. An immense weight or mass may be raised through a small space, by first stretching a dry rope between it and a support, and then wetting the rope.*

RESISTANCE OF FLUIDS.

441. The Resistance to a body moving in a fluid, arises from the inertia, from the cohesion, and from the friction of the fluid, admitting the particles to be in contact. The influence of this last cause, granting it to exist, is probably very small; and the second is in most fluids inconsiderable, when compared with the inertia. The resistance, therefore, which we shall here consider, is that which arises from the Inertia of the fluid.†

442. *The resistance which a plane surface meets with while it moves in a fluid, in a direction perpendicular to its plane, is proportioned to the square of its velocity.*

Whatever motion or momentum is imparted to the fluid, exactly the same amount is extinguished in the moving body, constituting the resistance (R). But the momentum is proportioned to the quantity of matter and velocity conjointly; or $M \propto Q \times V$. Again, in the present case, the quantity of fluid displaced must evidently be proportioned to the velocity of the moving body; that is, $Q \propto V$. Therefore, M or $R \propto V^2$.

This proposition is found to hold good in practice, where the velocity is very small, as the motions of boats or vessels in water; but when the velocity becomes very great, as that of a cannon ball, the resistance increases in a much higher ratio than as the square of the

* Arnott's El. Phys. I. 19.—Robison's Mech. Phil. † Vince's Hydrostatics.

velocity. (See Art. 357.) Since action and reaction are equal, it makes no difference, in the foregoing proposition, whether we consider the plane in motion and the fluid at rest, or the fluid in motion and striking against the plane at rest.

443. On account of the rapidity with which the resistance increases as the velocity is augmented, when a vessel or a steamboat is moving in water, it is only a comparatively moderate velocity that can possibly be given to it. A vessel driven by a wind which moves 60 miles an hour, is not carried forward faster than at the rate of 12 or 14 miles per hour. Steamboats are sometimes urged forward at the rate of 16 miles an hour; but to gain the additional speed over and above 12 miles, requires a great expenditure of force. If a steam engine of 20 horse power give a motion of 4 miles an hour, it would require one of 180 horse power to increase the speed to 12 miles an hour. But, it must be observed that the resistance decreases as fast when the velocity is diminished, as it increases when the velocity is augmented; and consequently, that canals may have the advantage over railways, when heavy articles are to be transported by very slow motions, although railways, encountering only the resistance of the air instead of water, have greatly the advantage when the motion is swift.*

Reckoning the resistance to increase only as the square of the velocity, it follows that twice the speed encounters four times the resistance; four times the speed, sixteen times; and ten times the speed, one hundred times the resistance. Hence a body descending freely through the air by gravity, for a great distance, does not continue to be accelerated throughout the whole distance, but is finally brought, by the resistance of the air, to a uniform motion.

Notwithstanding the difficulties attending the mathematical theory of Hydraulics, so much has already been done by the assistance of practical investigations, that we may in general, by comparing the results of former experiments with our calculations, predict the effect of any proposed arrangement, without an error of more than one fifth, or perhaps one tenth, of the whole,—a degree of accuracy fully sufficient for practice.†

* See Leslie, Nat. Phil. I. 443. † Dr. Young's Lectures on Nat. Phil. I. 277.

Formation of Waves.

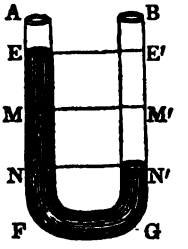
444. When the surface of water is pressed upon unequally, in parts contiguous to one another, the columns most pressed are shortened, and sink beneath the natural level of the surface, while those that are least pressed are lengthened, and rise above that level. As soon as the former columns have sunk to a certain depth, and the latter have risen to a certain height, their motions are reversed, and continue so, until the columns that were at first most depressed have become most elevated, and those that were most elevated have become most depressed.*

445. *The alternate elevations and depressions of the surface of a body of water, produced by a force acting unequally on the surface, are called waves.*

The water in the formation of waves has a vibratory or reciprocating motion, both in a horizontal and in a vertical direction, by which it passes from the columns that are shortened to those that are lengthened, and returns again in the opposite direction. *Progressive* motion is not necessary to undulation.†

446. Sir Isaac Newton first observed the analogy between the motions of waves and the vibrations of a column of water in a recurved tube, and upon this analogy he founded his theory of waves. Let AFGB (Fig. 185.) be a bent tube, of equal bore throughout, having its sides parallel to each other and perpendicular to the horizon. Suppose it to be filled with water or any fluid to the height MM′. By any pressure applied at M′, let the column be depressed to N′

Fig. 185.



* Playfair's Outlines, I. 203. † Playfair.

and raised to E in the opposite arm. The pressure being removed, the longer column EF will preponderate and seek to regain its original level, but the ascending column will not stop at M′, but on account of its inertia, will ascend to E′, that is, to the same height as that from which it descended on the other side. It will now descend again, and these reciprocal motions will continue until destroyed by the natural impediments to motion. On account of these, each successive vibration is shorter than the preceding, but all of them, like those of a pendulum, are performed in equal times; for the moving force is obviously proportioned to the column EM, that is, to the space through which the whole column vibrates; and when the forces are as the spaces, the times are equal.

447. Now when the surface of water is smooth and at rest, if any force (as the action of the wind or the fall of a stone) depress that surface in any particular place, the contiguous water will necessarily rise all around that place. The water which has thus been elevated, descends soon after in consequence of its gravity; and by the time it has reached the original level, it will have acquired velocity sufficient to carry it lower than that level; therefore, it now acts as another original moving force, in consequence of which, the water will be raised on both sides of it. And for the same reason, the descent of those elevated parts will produce other elevations contiguous to them, and so on. Thus the alternate rising and falling of the water in ridges, will expand all around the original place of motion; but as they recede from that place, so the ridges, as well as the adjoining hollows, grow smaller and smaller until they vanish. This diminution of size is produced by three causes, namely, by the want of perfect freedom of motion amongst the particles of water, by the resistance of the air, and by the remoter ridges being larger in diameter than those which are nearer.

448. From a variety of experiments and observations, it appears that the utmost force of the wind cannot penetrate a great way into the water; and that even in violent storms the water of the sea is slightly agitated at the depth of twenty feet below the usual level, and probably not moved at all at the depth of thirty feet.* Therefore,

* Boyle's Works, Vol. III. in Cavallo, I. 260.

the actual displacing of the water by the wind cannot be supposed to reach nearly so low; and hence it would seem that the greatest waves could not be so very high as they are often represented by navigators. But it must be observed that in storms waves increase to an enormous size from the *accumulation of waves upon waves*; for, as the wind is continually blowing, its action will raise a wave upon another wave, and a third wave upon a second, in the same manner as it raises a wave upon the flat surface of the water. In fact, at sea, a variety of waves of different sizes are frequently seen one upon the other, especially while the wind is actually blowing. When it blows fresh, the tops of the waves, being lighter and thinner than the other parts, are impelled forward, broken, and turned into a white foam, particles of which, called *spray*, are carried to a great distance.*

449. Whilst the depth of the water is sufficient to allow the oscillation to proceed undisturbed, the waves have no progressive motion, and are kept, each in its place, by the action of the waves that surround it. But if, by a rock rising near to the surface, or by the shelving of the shore, the oscillation is prevented, or much retarded, the waves in the deep water are not balanced by those in the shallower, and therefore acquire a progressive motion in this last direction, and form *breakers*. Hence it is that waves always break against the shore, whatever be the direction of the wind. Breakers formed over a great extent of shore, are distinguished by the name of *surf*. The surf is greatest in those parts of the earth where the wind blows always nearly in the same direction.†

* Cavallo. † Playfair.

APPENDIX.

Illustrations of Guldinus's Properties of the Center of Gravity. (Vol. I. p. 223.)

In any figure, the first property holds good when applied to any one of the bounding lines, or to any number, or to the whole.

First.—Let ABC (Fig. 186.) be a TRIANGLE, forming a cone by its revolution on the line AB.

1. *For the line* AC.

Bisect AC in E, its center of gravity, and draw DE parallel to BC. Then in the revolution of the triangle, AC will describe the convex surface of the cone, and E will describe a circle whose radius is DE, and circumference (the path described by the center of gravity of AC) $=2\pi DE=\pi BC$ $\therefore$

$$2\pi DE \times AC=\pi BC \times AC=$$ *convex surface of the cone.*

2. *For the line* BC.

Bisect BC in F, which is its center of gravity. In the revolution of the triangle, BC will describe the circular base of the cone, and the point F will describe the circumference of a circle, which equals $2\pi BF=\pi BC$ $\therefore$

$$2\pi BF \times BC=\pi BC^2=$$ *surface of the base.*

3. *For the two lines* AC *and* BC.

Since AC and BC may be considered as collected in the points E and F respectively, their common center of gravity is in the line EF. Let it be at G, and draw KG parallel to BC. Then the circumference described by $G=2\pi KG=\pi BC$ $\therefore$

$2\pi KG(AC+BC)=\pi BC(AC+BC)=\pi BC \times AC+\pi BC^2=$ *convex surface and base.*

4. *For the three sides* AB, BC, AC.

Since G is the common center of gravity of AC and BC, and D that of AB, the common center of gravity of the three lines is in DG. Let it be at G′, and through G′ draw HG′I parallel to BC.

Then the path of the center of gravity will be the circumference of the circle whose radius is HG′.

Put $a=\frac{1}{2}$AC, $b=\frac{1}{2}$BC, $c=\frac{1}{2}$AB. Then

$a+b : c :: DG' : G'G :: HG' : G'I$. ∴

$$a+b+c : c :: HG'+G'I(b) : G'I=\frac{bc}{a+b+c}$$ ∴

$$HG'=b-\frac{bc}{a+b+c}=\frac{ab+b^2}{a+b+c}.$$ *cir.* $HG'=2\pi\left(\frac{ab+b^2}{a+b+c}\right)$ ∴

cir. $HG' \times 2(a+b+c)=4\pi ab+4\pi b^2=\pi BC \times AC+\pi BC^2=$ ***both surfaces as before.***

Fig. 186.

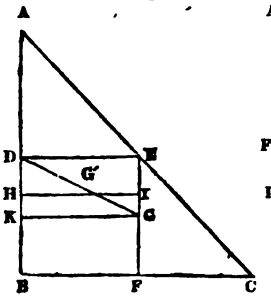


Fig. 187.

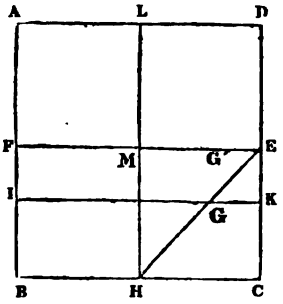
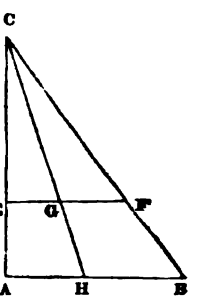


Fig 188.



Secondly.—Let ABCD, (Fig. 187.) be a PARALLELOGRAM, forming a *cylinder* by its revolution on AB.

1. *For the line* DC.

Bisect DC in E, and draw FE parallel to BC. Then E is the center of gravity which, in its revolution, describes the circle of which FE (=BC) is the radius. Therefore,

cir. FE × DC = *cir.* BC × DC = *convex surface.*

2. *For the line* BC *or* AD.

Bisect BC in H, then H is the center of gravity which describes, in the revolution, the circumference of a circle of which BH is the radius. Therefore, *cir.* $BH=\frac{1}{2}$ *cir.* BC ∴

cir. $BH \times BC=\frac{1}{2}$ *cir.* $BC \times BC=\frac{1}{2}\pi 2BC \times BC=\pi BC^2=$ ***base.***

3. *For the lines* BC *and* CD.

Join HE, and let G be the common center of gravity of the two lines, and through G draw the line IGK parallel to BC.

Put $EC=a$, $HC=b$, then $HE=\sqrt{(a^2+b^2)}$.

$a : b : \mathrm{HG} : \mathrm{GE}$, and $a+b : b :: \sqrt{(a^2+b^2)} : \mathrm{GE} = \frac{b\sqrt{(a^2+b^2)}}{a+b}$ ∴

$\sqrt{(a^2+b^2)} : b :: \frac{b\sqrt{(a^2+b^2)}}{a+b} : \mathrm{GK} = \frac{b^2}{a+b}$ ∴

$\mathrm{IG} = 2b - \frac{b^2}{a+b} = \frac{2ab+b^2}{a+b}$ ∴ *cir.* $\mathrm{IG} = 2\pi\left(\frac{2ab+b^2}{a+b}\right)$.

cir. $\mathrm{IG} \times (\mathrm{BC}+\mathrm{DC}) = 2\pi\left(\frac{2ab+b^2}{a+b}\right) \times 2(a+b) = 4\pi(2ab+b^2) =$

$8\pi ab + 4\pi b^2 =$ *convex surface and base.*

4. *For the three lines* AD, BC *and* DC.

Through E draw EF and bisect it in M, then M is the common center of gravity of AD and BC. Since E is the center of gravity of DC, the common center of gravity of the three lines is in ME. Let it be in G′. Then, $a : 2b :: \mathrm{G'M} : \mathrm{G'E}$, and

$a+2b : 2b : \mathrm{G'M}+\mathrm{G'E}(b) : \mathrm{G'E} = \frac{2b^2}{a+2b}$, ∴

$\mathrm{FG'} = 2b - \frac{2b^2}{a+2b} = \frac{2ab+2b^2}{a+2b} = \frac{2(ab+b^2)}{a+2b}$ ∴

cir. $\mathrm{FG'} \times (\mathrm{AD}+\mathrm{DC}+\mathrm{BC}) = 4\pi\frac{(ab+b^2)}{a+2b} \times 2(a+2b) = 8\pi(ab+b^2)$

$=$ *convex surface and two circular ends.*

5. *For the four lines* AB, BC, CD, DA.

The common center of gravity of all these lines is at M. Therefore,

cir. $\mathrm{FM} = \frac{1}{2}$ *cir.* $\mathrm{BC} =$ *cir.* b ∴

cir. $\mathrm{FM} \times (\mathrm{AB}+\mathrm{BC}+\mathrm{CD}+\mathrm{DA}) = 2\pi b(4a+4b) = 8\pi(ab+b^2) =$

convex surface and two circular ends as before.

Examples of the second property in the solid cone and cylinder.

1. *For the cone.*

Let ABC (Fig. 188.) be a triangle. Bisect AB in H, join CH and and take $\mathrm{CG} = \frac{2}{3}\mathrm{CH}$, and G is the center of gravity of the triangle. Through G draw EGF parallel to AB. Then $\mathrm{EG} = \frac{1}{3}\mathrm{AB}$ ∴ $2\pi\mathrm{EG} = \frac{2}{3}\pi\mathrm{AB}$. But $\mathrm{AB} \times \frac{1}{2}\mathrm{AC} =$ area of the triangle ∴

$2\pi\mathrm{EG} \times \mathrm{AB} \times \frac{1}{2}\mathrm{AC} = \frac{2}{3}\pi\mathrm{AB} \times \mathrm{AB} \times \frac{1}{2}\mathrm{AC} = \frac{\pi\mathrm{AB}^2 \times \mathrm{AC}}{3} =$ *solidity of the cone.*

2. *For the cylinder.*

In figure 187, M is the center of gravity of the generating rectangle. Then $FM = \frac{1}{2}BC$.·.

$2\pi FM = \pi BC$. But $BC \times AC$ = surface of the rectangle .·.

$2\pi FM \times BC \times AC = \pi BC \times BC \times AC = \pi BC^2 \times AC$ = *solidity of the cylinder.*

END OF VOL. I.

